Exo 4 Let (u_n) be a real sequence defined by: $\forall n \in \mathbb{N}: u_{n+1} = 1 + \frac{1}{u_n}$ and $u_0 = 1$.

① Prove that: $\forall n \in \mathbb{N}: u_n \geq 1$.

② We denote "a" as the positive solution of the equation: $x = 1 + \frac{1}{x}$.

ⓐ Prove that: $\forall n \in \mathbb{N}$: $|u_{n+1} - a| \leq \frac{1}{a}|u_n - a|$ and $|u_n - a| \leq \frac{1}{a^n}|u_0 - a|$.

ⓑ What do you conclude?

Solution:

① We use induction to prove that: $\forall n \in \mathbb{N}: u_n \geq 1$.

Step1 (Base Case: $n = 1$)

Starting with $u_0 = 1 \geq 1$, which is clearly true.

Step2 (Inductive hypothesis)

Assume that $u_n \geq 1$ is true for some $n \in \mathbb{N}$.

Step3 (Inductive step)

We need to prove that, if $u_n \geq 1$ is true for $n \in \mathbb{N}$, it is also true for $n+1$.

We know that: $u_n \geq 1 \Rightarrow 0 < \frac{1}{u_n} \leq 1$

$$\Rightarrow 1 < \frac{1}{u_n} + 1 \leq 2$$

$$\Rightarrow u_{n+1} > 1$$

$$\Rightarrow u_{n+1} \geq 1, \text{ as required.}$$

② The positive solution of $x = 1 + \frac{1}{x}$ is the golden number $a = \varphi = \frac{1+\sqrt{5}}{2} > 1$.

ⓐ We have $|u_{n+1} - a| = \left|1 + \frac{1}{u_n} - 1 - \frac{1}{a}\right| = \left|\frac{a - u_n}{a u_n}\right|$

$$= \frac{1}{a u_n}|u_n - a|,$$

Since $u_n \geq 1 \Rightarrow a u_n \geq a$

$$\Rightarrow \frac{1}{a u_n} \leq \frac{1}{a}$$

thus $|u_{n+1} - a| \leq \frac{1}{a}|u_n - a|$ as required

Now, set $P(n) \equiv |u_n - a| \leq \frac{1}{a^n} |u_0 - a|$;

we use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

for $n = 0$, we have : $(P(0) \equiv) \; |u_0 - a| = \frac{1}{a^0} |u_0 - a|$

$$\Rightarrow |u_0 - a| \leq \frac{1}{a^0} |u_0 - a|$$

$$\Rightarrow P(0) \text{ is true}.$$

Assume that $P(n)$ is true for some $n \in \mathbb{N}$, we seek to demonstrate that $P(n+1)$ is also true.

Now, we have : $|u_{n+1} - a| \leq \frac{1}{a} |u_n - a|$

$$\Rightarrow |u_{n+1} - a| \leq \frac{1}{a} \frac{1}{a^n} |u_0 - a| \quad \text{(by assumption that } P(n) \text{ is true)}$$

$$\Rightarrow |u_{n+1} - a| \leq \frac{1}{a^{n+1}} |u_0 - a|$$

$$\Rightarrow P(n+1) \text{ is true}$$

that is $\forall n \in \mathbb{N} : |u_n - a| \leq \frac{1}{a^n} |u_0 - a|$ as required

(b) conclusion :

Since $\forall n \in \mathbb{N} : \; 0 < |u_n - a| \leq \frac{1}{a^n} |u_0 - a|$ and $\lim_{n \to +\infty} \frac{1}{a^n} |u_0 - a| = 0$,

by squeeze theorem, we conclude that : $\lim_{n \to +\infty} |u_n - a| = 0$

therefore $\lim_{n \to +\infty} u_n = a$.

Exo5 : (1) Prove that if the two subsequences (u_{2n}) and (u_{2n+1}) converge towards l, then the sequence (u_n) also converges towards l.

(2) Application : let (u_n) be a sequence where : $\forall p \in \mathbb{N}^*, \forall n \in \mathbb{N}^* : 0 \leq u_{n+p} \leq \frac{1}{n}$

- Prove that (u_n) converges towards 0.

(3) Application : let (v_n) be a decreasing and converging sequence towards 0, and let (S_n) be defined by $S_n = \sum_{i=0}^{n} (-1)^i v_i$.

- Prove that the two subsequences (S_{2n}) and (S_{2n+1}) are adjacent. What do you conclude?

Solution:

① We know:

$\lim u_{2n} = l \iff \forall \varepsilon>0, \exists N_1 \in \mathbb{N}, \forall n \in \mathbb{N}: (n>N_1 \Rightarrow |u_{2n}-l|<\varepsilon) \cdots ①$

$\lim u_{2n+1} = l \iff \forall \varepsilon>0, \exists N_2 \in \mathbb{N}, \forall n \in \mathbb{N}: (n>N_2 \Rightarrow |u_{2n+1}-l|<\varepsilon) \cdots ②$

Now, let any $\varepsilon>0$, it suffice to choose $N=\max\{2N_1, 2N_2+1\}$ to get:

$$\forall n \in \mathbb{N}: (n>N \Rightarrow |u_n-l|<\varepsilon)$$

Indeed;

If $n=2k$, we obtain: $2k>N \Rightarrow 2k>2N_1$

$\Rightarrow k>N_1$

$\Rightarrow |u_{2k}-l|<\varepsilon$ (from ①)

$\Rightarrow |u_n-l|<\varepsilon$

If $n=2k+1$, We get: $2k+1>N \Rightarrow 2k+1>2N_2+1$

$\Rightarrow k>N_2$

$\Rightarrow |u_{2k+1}-l|<\varepsilon$ (from ②)

$\Rightarrow |u_n-l|<\varepsilon$

That is, $\forall n \in \mathbb{N}: (n>N \Rightarrow |u_n-l|<\varepsilon)$

as required.

② Application: We have the sequence (u_n) defined by:

$$\forall p \in \mathbb{N}^*, \forall n \in \mathbb{N}^*: 0 \le u_{n+p} \le \frac{1}{n}+\frac{1}{p} \quad \cdots ①$$

Putting $p=n$ then $p=n+1$ in the previous inequality ①, we get:

$$\forall n \in \mathbb{N}^*: 0 \le u_{2n} \le \frac{2}{n}$$

and $\forall n \in \mathbb{N}^*: 0 \le u_{2n+1} \le \frac{1}{n}+\frac{1}{n+1}$ respectively.

Since $\lim\left(\frac{2}{n}\right)=0$ and $\lim\left(\frac{1}{n}+\frac{1}{n+1}\right)=0$, we find that: $\lim u_{2n} = \lim u_{2n+1} = 0$,

therefore, (u_n) converges to 0.

③ Application: We have the sequence (a_n) decreasing and converging to 0, and (S_n) is defined by: $S_n = \sum_{i=0}^{n} (-1)^i a_i$.

First of all, because (a_n) is decreasing, we find that:

$$S_{2(n+1)} - S_{2n} = S_{2n+2} - S_{2n}$$
$$= \sum_{i=0}^{2n+2} (-1)^i v_i - \sum_{i=0}^{2n} (-1)^i v_i$$
$$= \sum_{i=2n+1}^{2n+2} (-1)^i v_i = -v_{2n+1} + v_{2n+2} \leq 0 \quad (\text{because } (v_n) \searrow)$$

$$\Rightarrow (S_{2n}) \searrow$$

also $S_{2(n+1)+1} - S_{2n+1} = S_{2n+3} - S_{2n+1}$
$$= \sum_{i=0}^{2n+3} (-1)^i v_i - \sum_{i=0}^{2n+1} (-1)^i v_i$$
$$= \sum_{i=2n+2}^{2n+3} (-1)^i v_i = v_{2n+2} - v_{2n+3} \geq 0$$

$$\Rightarrow (S_{2n+1}) \nearrow$$

Secondly: Since (v_n) converges to 0, then: $\lim v_{2n+1} = 0$

thus, $\lim (S_{2n+1} - S_{2n}) = \lim v_{2n+1} = 0$

therefore, the sequences (S_{2n}) and (S_{2n+1}) are adjacent

hence, the sequence (S_n) converges. ~~[illegible]~~

Exo6: Let a, b be real numbers, where $0 < a < b$, We define the two sequences (u_n) and (v_n) as follows:

$$\forall n \in \mathbb{N}: u_{n+1} = \sqrt{u_n v_n}, \quad v_{n+1} = \frac{u_n + v_n}{2}, \quad v_0 = b, \; u_0 = a.$$

- Prove the following: ① $\forall n \in \mathbb{N}: 0 < u_n < v_n$

② The two sequences (u_n) and (v_n) are monotonic.

③ $\forall n \in \mathbb{N}: v_{n+1} - u_{n+1} \leq \frac{1}{2}(v_n - u_n)$.

④ $\forall n \in \mathbb{N}: v_n - u_n \leq \left(\frac{1}{2}\right)^n (b-a)$.

⑤ $\lim (v_n - u_n) = 0$, what do you conclude?

Solution: ① First, we have $0 < a < b \Rightarrow 0 < u_0 < v_0$.

Secondly, we assue that: $0 < u_n < v_n$ and we have to prove that:

$0 < u_{n+1} < v_{n+1}$.

Now, we know that: $u_{n+1} = \sqrt{u_n v_n} > 0$
on the other hand, we have:

$$v_{n+1} - u_{n+1} = \frac{u_n + v_n}{2} - \sqrt{u_n v_n}$$
$$= \frac{1}{2}\left(u_n + v_n - 2\sqrt{u_n}\sqrt{v_n}\right)$$
$$= \frac{1}{2}\left(\sqrt{u_n} - \sqrt{v_n}\right)^2 > 0$$

$\Rightarrow 0 < u_{n+1} < v_{n+1}$, as required

② We have, for any $n \in \mathbb{N}$: $v_{n+1} - v_n = \frac{u_n + v_n}{2} - v_n = \frac{u_n - v_n}{2} < 0$

$\Rightarrow (v_n) \downarrow$

also, for any $n \in \mathbb{N}$: $u_{n+1} - u_n = \sqrt{u_n v_n} - u_n = \sqrt{u_n}\left(\sqrt{v_n} - \sqrt{u_n}\right) > 0$

$\Rightarrow (u_n) \uparrow$

③ Let any $n \in \mathbb{N}$; we have:

$$v_{n+1} - u_{n+1} < v_{n+1} - u_n \quad (\text{because } u_{n+1} > u_n)$$

$$\Rightarrow v_{n+1} - u_{n+1} < \frac{u_n + v_n}{2} - u_n$$

$$\Rightarrow v_{n+1} - u_{n+1} < \frac{1}{2}(v_n - u_n).$$ as desired.

④ Reasoning by induction:

First, we have: $v_0 - u_0 = b - a \leq \left(\frac{1}{2}\right)^0 (b-a)$ is true

Secondly, we assume that: $v_n - u_n \leq \left(\frac{1}{2}\right)^n (b-a)$ is true for some $n \in \mathbb{N}$, and we try to prove that $v_{n+1} - u_{n+1} \leq \left(\frac{1}{2}\right)^{n+1} (b-a)$ is also true.

From question ③, we have $v_{n+1} - u_{n+1} \leq \frac{1}{2}(v_n - u_n)$

thus $v_{n+1} - u_{n+1} \leq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n (b-a)$ (from the assumption of induction)

Therefore $v_{n+1} - u_{n+1} \leq \left(\frac{1}{2}\right)^{n+1} (b-a)$

as required.

⑤ Since $\lim \left(\frac{1}{2}\right)^n (b-a) = 0$,

then, we have $\lim (v_n - u_n) = 0$.

We conclude that (u_n) and (v_n) are adjacent.

No 7

① Let (u_n) be a real sequence, where: $\forall n \in \mathbb{N}: |u_{n+1} - u_n| \leq a^n$ $(0<a<1)$

- Prove that (u_n) is a Cauchy sequence.

Solution:

We have to prove that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \forall p \in \mathbb{N}: (n > N \Rightarrow |u_{n+p} - u_n| < \varepsilon$$

Let any $\varepsilon > 0$, we're looking for a suitable $N \in \mathbb{N}$ s.t.

$$\forall n, p \in \mathbb{N}: (n > N \Rightarrow |u_{n+p} - u_n| < \varepsilon)$$

Now, we have:

$$|u_{n+p} - u_n| = |u_{n+p} - u_{n+p-1} + u_{n+p-1} - u_{n+p-2} + \cdots + u_{n+1} - u_n|$$

$$\leq |u_{n+p} - u_{n+p-1}| + |u_{n+p-1} - u_{n+p-2}| + \cdots + |u_{n+1} - u_n|$$

$$\leq a^{n+p-1} + a^{n+p-2} + \cdots + a^n = a^n \frac{1-a^p}{1-a}$$

$$\leq \frac{a^n}{1-a} \quad \cdots\cdots \quad (1)$$

Since $\lim\limits_{n \to +\infty} \left(\frac{a^n}{1-a}\right) = 0$, we can find $N \in \mathbb{N}$, s.t.

$$\forall n \in \mathbb{N}: n > N \Rightarrow \frac{a^n}{1-a} < \varepsilon$$

$$\Rightarrow |u_{n+p} - u_n| < \varepsilon$$

(from the inequality ①)

thus, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, p \in \mathbb{N}: (n > N \Rightarrow |u_{n+p} - u_n| < \varepsilon)$

as required