Series of exercises 3 (questions marked * left to the students)

Exercise 01

Using the definition, prove that:

1) $\lim\limits_{x\to3}\frac{x^2-1}{x^2+1}=\frac{4}{5}$, *2)* $\lim\limits_{x\to\infty}\frac{x^2-1}{x^2+1}=1$, *3)* $\lim\limits_{x\to1}\frac{x+2}{(x-1)^2}=+\infty$,

*4)** $\lim\limits_{x\to-1}\frac{x+3}{x+2}=2$, *5)** $\lim\limits_{x\to\infty}\frac{2x^2+x+1}{x^2-3x}=2$, *6)** $\lim\limits_{\substack{> \\ x\to1}}\frac{2x^2-x-2}{x^2-x}=-\infty$.

Exercise 02 Calculate the following limits:

1) $\lim\limits_{x\to1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$, *2*)* $\lim\limits_{x\to1}\left(\frac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}\right)$ *4)* $\lim\limits_{x\to\mp\infty}\left(\sqrt{x^2-2x}-\sqrt{x^2-1}\right)$,

5) $\lim\limits_{x\to\mp\infty}x\ln\frac{x}{x+2}$, *6)** $\lim\limits_{x\to\mp\infty}x\ln\frac{x^2+x}{x^2+2x+3}$, *7*)* $\lim\limits_{x\to\mp\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2}$,

8) $\lim\limits_{x\to+\infty}\left(\frac{x+3}{x-2}\right)^{2x+1}$, *9)* $\lim\limits_{x\to\pm\infty}\frac{x}{E(x)+1}$, *10)** $\lim\limits_{x\to\pm\infty}xE(x)$,

*11)** $\lim\limits_{x\to+\infty}\frac{x+\cos(x^2+2x)}{x^2-2}$.

Remark: For questions 5,6,7 and 8, we use the limit: $\lim\limits_{x\to0}\frac{\ln(1+x)}{x}=1$.

Exercise 04

Prove that the function f does not have a limit at x_0 in each of the following cases:

1) $x_0=0$ and $f(x)=\sin\frac{1}{x}$, *2)** $x_0=\infty$ and $f(x)=\cos x$,

*3)** $x_0=0$, $f=v\circ u$, $u(x)=x\cos\frac{1}{x}$ and $v(x)=\begin{cases}1, & x=0\\ 0, & x\neq0\end{cases}$.

Exercise 05

Can the function f be continuously extended at x_0 in each of the following cases:

1) $f(x)=\frac{x^3+5x+6}{x^5+1}$ and $x_0=-1$, *2)** $f(x)=\frac{x^2-x}{\sin(x-1)}$ and $x_0=1$,

*3)** $f(x)=1-x\sin\frac{1}{x}$ and $x_0=0$.

Exercise 06

Determine a and b such that the function f is continuous in its domain.

1) $f(x)=\begin{cases}x & si\ |x|\leq1\\ x^2+ax+b & si\ |x|>1\end{cases}$, *2)** $f(x)=\begin{cases}(x-1)^3 & si\ x\leq0\\ ax+b & si\ 0<x<1\\ \sqrt{x} & si\ x>1\end{cases}$.

Exercise 07

Apply the Mean value theorem to prove the following:

1) $\forall x \in]0;1[: 1 + x < e^x < \frac{1}{1-x}$, 2)*$\forall x \geq 0 :\ 2 \leq \sqrt{4+x^2} \leq 2 + \frac{x^2}{2}$

3)* $na^{n-1}(b-a) < b^n - a^n < nb^{n-1}(b-a)$ where $0 < a < b$ and $n \geq 2$ be an integer.

Exercise 08 Calculate the following limits using L'Hôpital's rule:

1) $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$, 2) $\lim\limits_{x\to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\cos x}$, 3) $\lim\limits_{x\to +\infty} x\left(e^{\frac{2x+1}{x^2}} - 1\right)$, 4)* $\lim\limits_{x\to 0} \frac{\tan x - x}{x - \sin x}$

5)* $\lim\limits_{x\to +\infty} \frac{\ln(2x^3 - 3x^2 + x - 1)}{2x - 1}$, 6) $\lim\limits_{x\to +\infty} x \ln\left[\tan\left(\frac{\pi}{4} + \frac{\pi}{x}\right)\right]$, 7)* $\lim\limits_{x\to 1} \left(\frac{1}{lnx} - \pi \text{cotan}\, \pi x\right)$

8)* $\lim\limits_{x\to 0} \frac{3\tan 4x - 12\tan x}{3\sin 4x - 12\sin x}$, 9) $\lim\limits_{x\to \frac{\pi}{4}} \frac{2\sqrt{2} - (\cos x + \sin x)^3}{1 - \sin 2x}$, 10)* $\lim\limits_{x\to 0} \frac{1}{x} \ln\left(\frac{e^x - 1}{x}\right)$.

Exercise 09 Let the function f be defined on $\mathbb{R}$ as follows: $f(x) = \begin{cases} \frac{-x^2}{x+2} & si\ x \geq 0 \\ \ln\frac{2x^2+1}{x^2+1} & si\ \ x < 0 \end{cases}$.

1) Examine the continuity of f over $\mathbb{R}$.

2) Is the function f differentiable at 0?

3) Express $f'(x)$ in terms of x.

4) Prove that f has an reciprocal function f^{-1} This involves defining its definition domain and expressing $f^{-1}(x)$ in terms of x.

Exercise 10 * Let the function f be defined on $\mathbb{R}$ as follows $f(x) = \begin{cases} \frac{\sin ax}{x} & si\ x < 0 \\ e^{bx} - x\ \ si\ \ x \geq 0 \end{cases}$

where a and b are real numbers.

1) Determine a such that f is continuous at 0.

2) Determine the value of b so that f is differentiable at 0.

Exercise 11

I) Prove the following:

1)* $\forall x \in]-1,1[: \tan \text{Arc}\sin x = \frac{x}{\sqrt{1-x^2}}$.

2) $\forall x \geq 0 : Arc\tan(x+1) - \text{Arc}\tan x = \text{Arc}\tan\frac{1}{1+x+x^2}$.

3)* $\forall x \in [-1,1]: \text{Arc}\cos x + \text{Arc}\cos(-x) = \pi$.

II) Calculate the following limits using L'Hôpital's rule:

1) $\lim\limits_{x\to 0} \frac{x\,\text{Arc}\sin x^2}{x\cos x - \sin x}$) , 2)* $\lim\limits_{x\to 0} \left(\frac{1}{x\text{Arc}\tan x} - \frac{1}{x^2}\right)$, 3) $\lim\limits_{x\to 0} (e^x + x)^{\frac{1}{x}}$, 4) $\lim\limits_{x\to 0} \left(\tan\frac{\pi}{x+4}\right)^{\frac{1}{x}}$,

5)* $\lim\limits_{x\to \frac{\pi}{2}} (\tan x)^{2\cos x}$, 6)* $\lim\limits_{x\to 0} \frac{\cosh x - \cos x - x^2}{x^6}$.