Larbi Ben Mhidi University - Oum El Bouaghi-

Faculty of exact sciences, natural and life sciences

Department of Mathematics and Computer Science **Academic year: 2023/2024**

Level: L1 Computer S **Duration:1h30**

Catch-up exam: Analysis 1

Exercise 1 (06 pts) (6=3+3)

1) Let $A = \left\{\frac{n}{n+1}; n \in \mathbb{N}\right\}$, specify if possible (With justification) $\sup A$, $\inf A$, $\max A$, $\min A$.

2) a) Prove that: $e^{5ix} - e^{-5ix} = \left(e^{ix} - e^{-ix}\right)\left(e^{4ix} + e^{-4ix} + e^{2ix} + e^{-2ix} + 1\right)$.

b) Deduce that:

$$e^{4i\frac{\pi}{5}} + e^{-4i\frac{\pi}{5}} + e^{2i\frac{\pi}{5}} + e^{-2i\frac{\pi}{5}} + 1 = 0,$$

and

$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0.$$

Given: $\forall k \in \mathbb{N}; \forall x \in \mathbb{R}: e^{kix} + e^{-kix} = 2\cos kx$ and $\cos 2x = 2\cos^2 x - 1$.

Exercise 2 (08 pts) (8=2.5+3+2.5)

Let f be a function defined in the interval $I = \left[\frac{1}{3}, 1\right]$ by

$$f(x) = \frac{x^2 + x}{3x^2 + 1}$$

1) a) Prove that the function f is strictly increasing on I.

b) Deduce that if $x \in I$ then $f(x) \in I$.

2) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by: $\forall n \in \mathbb{N}: \begin{cases} u_0 = 1 \\ u_{n+1} = \frac{u_n^2 + u_n}{3u_n^2 + 1} \end{cases}$.

a) Calculate u_1, u_2 and prove that $\forall n \in \mathbb{N}: u_n > \frac{1}{3}$.

b) prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly decreasing.

c) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

3) a) Prove that: $\forall n \in \mathbb{N}: \left(u_{n+1} - \frac{1}{3}\right) \leq \frac{3}{4}\left(u_n - \frac{1}{3}\right)$ and deduce that

$$\forall n \in \mathbb{N}: \left(u_n - \frac{1}{3}\right) \leq \frac{2}{3}\left(\frac{3}{4}\right)^n$$

b) Deduce $\lim_{n\to\infty} u_n$.

Exercise 3 (06 pts) (6=4+2)

1) Using L'Hopital's rule, calculate the following limits

$$\lim_{x\to 0}\frac{\ln(1-x) + e^x - \cos x}{x^2} \quad , \quad \lim_{x\to\pm\infty}\left(x^2\sin\frac{1}{x} - x\right).$$

2) Applying the mean value theorem to the interval $[0, x]$, Prove that:

$$\forall x \in \mathbb{R}_+^*: 0 < \arctan x < x.$$

Good luck.

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Corrected catch-up exam: Analysis 1

Exercise 1 (06 pts)

1) Let $A = \left\{\frac{n}{n+1}; n \in \mathbb{N}\right\}$, specify if possible (With justification) $\sup A$, $\inf A$, $\max A$, $\min A$.

We have $\forall n \in \mathbb{N}: \frac{n}{n+1} = 1 - \frac{1}{n+1}$, so

$$\forall n \in \mathbb{N}: n+1 \geq 1 \Longrightarrow 0 \leq \frac{1}{n+1} \leq 1$$

$$\Longrightarrow -1 \leq -\frac{1}{n+1} \leq 0$$

$$\Longrightarrow 0 \leq 1 - \frac{1}{n+1} \leq 1.$$ (0.5)

For $n = 0$ then $\frac{n}{n+1} = 0$ so

$$\inf A = \min A = 0.$$ (2x0.5)

Let's prove that: $\sup A = 1$. Indeed

$$\sup A = 1 \Longleftrightarrow \begin{cases} \forall n \in \mathbb{N}: \frac{n}{n+1} \leq 1 \\ \forall \varepsilon > 0; \exists n_0 \in \mathbb{N}: 1 - \varepsilon < \frac{n_0}{n_0+1} \end{cases}$$

According to Archimedean axiom we have $\forall \varepsilon > 0. \exists n_0 \in \mathbb{N}^*: 1 < n_0\varepsilon < \varepsilon(n_0+1)$ so

$$1 < \varepsilon(n_0+1) \Longrightarrow -\varepsilon < -\frac{1}{n_0+1}$$

$$\Longrightarrow 1 - \varepsilon < 1 - \frac{1}{n_0+1}.$$

So

$$\forall \varepsilon > 0. \exists n_0 \in \mathbb{N}^*: 1 - \varepsilon < \frac{n_0}{n_0+1}.$$ (1)

we have $1 \notin A$ so $\max A =$ unvailable. (0.5)

2) a) Prove that: $e^{5ix} - e^{-5ix} = (e^{ix} - e^{-ix})(e^{4ix} + e^{-4ix} + e^{2ix} + e^{-2ix} + 1)$.

b) Deduce that: $e^{4i\frac{\pi}{5}} + e^{-4i\frac{\pi}{5}} + e^{2i\frac{\pi}{5}} + e^{-2i\frac{\pi}{5}} + 1 = 0$ and $4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$.

a) We have

$$(e^{ix}-e^{-ix})(e^{4ix}+e^{-4ix}+e^{2ix}+e^{-2ix}+1)=e^{ix}(e^{4ix}+e^{-4ix}+e^{2ix}+e^{-2ix}+1)$$
$$-e^{-ix}(e^{4ix}+e^{-4ix}+e^{2ix}+e^{-2ix}+1)$$
$$=e^{5ix}+e^{-3ix}+e^{3ix}+e^{-ix}+e^{ix}-e^{3ix}-e^{-5ix}-e^{ix}-e^{-3ix}-e^{-ix}$$
$$=e^{5ix}-e^{-5ix}.$$ (1)

b) by putting $x=\frac{\pi}{5}$ we get

$$\left(e^{i\frac{\pi}{5}}-e^{-i\frac{\pi}{5}}\right)\left(e^{4i\frac{\pi}{5}}+e^{-4i\frac{\pi}{5}}+e^{2i\frac{\pi}{5}}+e^{-2i\frac{\pi}{5}}+1\right)=e^{i\pi}-e^{-i\pi}=0.$$ (0.5)

Since $e^{i\frac{\pi}{5}}-e^{-i\frac{\pi}{5}}\neq 0$ we obtain

$$e^{4i\frac{\pi}{5}}+e^{-4i\frac{\pi}{5}}+e^{2i\frac{\pi}{5}}+e^{-2i\frac{\pi}{5}}+1=0.$$ (0.5)

And since $e^{4i\frac{\pi}{5}}+e^{-4i\frac{\pi}{5}}=2\cos\frac{4\pi}{5}$; $e^{2i\frac{\pi}{5}}+e^{-2i\frac{\pi}{5}}=2\cos\frac{2\pi}{5}$ we get

$$2\cos\frac{4\pi}{5}+2\cos\frac{2\pi}{5}+1=0$$ (0.5)

and since $\cos\frac{4\pi}{5}=2\cos^2\frac{2\pi}{5}-1$ we obtain

$$4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0.$$ (0.5)

Exercise 2 (08 pts)

Let f be a function defined in the interval $I=\left[\frac{1}{3},1\right]$ by $f(x)=\frac{x^2+x}{3x^2+1}$.

1) a) Prove that the function f is strictly increasing on I.

$$f'(x)=\frac{-3x^2+2x+1}{(3x^2+1)^2}.$$ (1)

$\Delta=16$; $x_1=-\frac{1}{3}$; $x_2=1$ so $\forall x\in I$: $f'(x)>0\Longrightarrow f$ is strictly increasing on I. (2x0.5)

b) Deduce that if $x\in I$ then $f(x)\in I$.

We have

$$x\in I\Longrightarrow \frac{1}{3}\leq x\leq 1$$
$$\xRightarrow{f \text{ is strictly increasing}} f\left(\frac{1}{3}\right)\leq f(x)\leq f(1)$$
$$\Longrightarrow \frac{1}{3}\leq f(x)\leq\frac{1}{2}\leq 1.$$ (0.5)

2) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by: $\forall n\in\mathbb{N}$: $\begin{cases}u_0=1\\ u_{n+1}=\frac{u_n^2+u_n}{3u_n^2+1}.\end{cases}$

a) Calculate u_1, u_2 and prove that $\forall n\in\mathbb{N}$: $u_n>\frac{1}{3}$.

$$u_1=\frac{1}{2};\ u_2=\frac{3}{7}.$$ (2x0.5)

$u_0=1>\frac{1}{3}$, suppose that $u_n>\frac{1}{3}$

$$u_n>\frac{1}{3}\xRightarrow{f\text{ is strictly increasing}} f(u_n)>f\left(\frac{1}{3}\right)\Rightarrow u_{n+1}>\frac{1}{3}.$$ (0.75)

b) prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly decreasing.

We have $u_0=1>\frac{1}{2}=u_1$ and f is strictly increasing so (u_n) is strictly decreasing. (0.75)

c) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

Since (u_n) is strictly decreasing and bounded a lower then the sequence (u_n) is covergent. (0.5)

3) a) Prove that: $\forall n\in\mathbb{N}$: $\left(u_{n+1}-\frac{1}{3}\right)\leq\frac{3}{4}\left(u_n-\frac{1}{3}\right)$.

We have $\forall n\in\mathbb{N}$:

$$\left(u_{n+1}-\frac{1}{3}\right)=\frac{u_n^2+u_n}{3u_n^2+1}-\frac{1}{3}=\frac{1}{3}\frac{3u_n-1}{3u_n^2+1}=\frac{1}{3u_n^2+1}\left(u_n-\frac{1}{3}\right).$$

On the other hand we have:

$$u_n>\frac{1}{3}\Rightarrow 3u_n^2+1>\frac{4}{3}$$
$$\Rightarrow\frac{1}{3u_n^2+1}<\frac{3}{4}$$
$$\Rightarrow\frac{1}{3u_n^2+1}<\frac{3}{4}$$
$$\Rightarrow\frac{1}{3u_n^2+1}\left(u_n-\frac{1}{3}\right)\leq\frac{3}{4}\left(u_n-\frac{1}{3}\right).$$

So

$$\left(u_{n+1}-\frac{1}{3}\right)\leq\frac{3}{4}\left(u_n-\frac{1}{3}\right).$$ (1.25)

Deduce that $\forall n\in\mathbb{N}$: $\left(u_n-\frac{1}{3}\right)\leq\frac{2}{3}\left(\frac{3}{4}\right)^n$.

$\left(u_0-\frac{1}{3}\right)\leq\frac{2}{3}\left(\frac{3}{4}\right)^0\Leftrightarrow\frac{2}{3}\leq\frac{2}{3}$, suppose that $\left(u_n-\frac{1}{3}\right)\leq\frac{2}{3}\left(\frac{3}{4}\right)^n$ so

$$\left(u_{n+1}-\frac{1}{3}\right)\leq\frac{3}{4}\left(u_n-\frac{1}{3}\right)$$
$$\leq\frac{3}{4}\left(u_n-\frac{1}{3}\right)$$
$$\leq\frac{3}{4}\frac{2}{3}\left(\frac{3}{4}\right)^n=\frac{2}{3}\left(\frac{3}{4}\right)^{n+1}.$$ (0.75)

b) Deduce $\lim\limits_{n\to\infty}u_n$.

We have $0 \leq \left(u_n - \frac{1}{3}\right) \leq \frac{2}{3}\left(\frac{3}{4}\right)^n$ since $\lim_{n\to\infty} \frac{2}{3}\left(\frac{3}{4}\right)^n = 0$ then $\lim_{n\to\infty} u_n = 0$. (0.5)

Exercise 3 (06 pts)

1) Using L'Hopital's rule, calculate $\lim_{x\to 0} \frac{\ln(1-x)+e^x-\cos x}{x^2}$, $\lim_{x\to\pm\infty} \left(x^2 \sin\frac{1}{x} - x\right)$.

$$\lim_{x\to 0} \frac{\ln(1-x)+e^x-\cos x}{x^2} = \left(I.F\,\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{(\ln(1-x)+e^x-\cos x)'}{(x^2)'}$$

$$= \lim_{x\to 0} \frac{\frac{-1}{1-x} + e^x + \sin x}{2x} = \left(I.F\,\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\left(\frac{-1}{1-x} + e^x + \sin x\right)'}{2(x)'}$$

$$= \lim_{x\to 0} \frac{\frac{-1}{(1-x)^2} + e^x + \cos x}{2}$$

$$= \frac{1}{2}. \qquad (2)$$

$$\lim_{x\to\pm\infty} \left(x^2 \sin\frac{1}{x} - x\right) = (I.F\ \infty.0).$$

So

$$\lim_{x\to\pm\infty} \left(x^2 \sin\frac{1}{x} - x\right) = \lim_{x\to\pm\infty} \frac{\sin\frac{1}{x} - \frac{1}{x}}{\frac{1}{x^2}} \left(I.F\,\frac{0}{0}\right)$$

$$= \lim_{x\to\pm\infty} \frac{\left(\sin\frac{1}{x} - \frac{1}{x}\right)'}{\left(\frac{1}{x^2}\right)'}$$

$$= \lim_{x\to\pm\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x} + \frac{1}{x^2}}{\frac{-2}{x^3}}$$

$$= \frac{1}{2}\lim_{x\to\pm\infty} \frac{\cos\frac{1}{x} - 1}{\frac{1}{x}} \left(I.F\,\frac{0}{0}\right)$$

$$= \frac{1}{2}\lim_{x\to\pm\infty} \frac{\left(\cos\frac{1}{x} - 1\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \frac{1}{2}\lim_{x\to\pm\infty} \frac{\frac{1}{x^2}\sin\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -\frac{1}{2} \lim_{x \to \pm\infty} \sin \frac{1}{x} = 0.$$ (2)

2) Applying the mean value theorem to the interval $[0, x]$, Prove that: $\forall x \in \mathbb{R}_+^*: 0 < \arctan x < x$.

We have $\forall x \in \mathbb{R}_+^*$ $\arctan x - \arctan 0 = \frac{1}{c^2+1}(x - 0)$ where $0 < c < x$,

so

$\forall x \in \mathbb{R}_+^*$ $\arctan x = \frac{1}{c^2+1} x$ where $0 < c < x$.

On the other hand we have:

$$c > 0 \Longrightarrow c^2 + 1 > 1$$

$$\Longrightarrow 0 < \frac{1}{c^2 + 1} < 1$$

$$\Longrightarrow 0 < \frac{1}{c^2 + 1} x < x$$

$$\Longrightarrow 0 < \arctan x < x.$$ (2)