Chapter Four: Real functions with real variable

4.1 Generalities

Definition 4.1

We call a real function of a real variable every application f of a subset D of $\mathbb{R}$ on set $\mathbb{R}$.

D is called the domain of definition for f.

We call the graph of the function f the subset of $\mathbb{R}^2$ which we denote by Γ_f, and defined as follows

$$\Gamma_f = \{(x; y) \in \mathbb{R}^2; x \in D \wedge y = f(x)\} \text{ or } \Gamma_f = \{(x; f(x)); x \in D\}.$$

The image of the domain D by f is denoted by $f(D)$ where: $f(D) = \{y \in \mathbb{R}; \exists x \in D: y = f(x)\}$.

Definition 4.2

Let $f: D \to \mathbb{R}$ be a function. We say that the function f is bounded from above (bounded from below, respectively) if, and only if, the set $f(D)$ is bounded from above (from below, respectively). So

$$(f \text{ is bounded from above}) \Leftrightarrow (\exists M \in \mathbb{R}; \forall x \in D: f(x) \leq M),$$

$$(f \text{ is bounded from below}) \Leftrightarrow (\exists m \in \mathbb{R}; \forall x \in D: f(x) \geq m)$$

We say that the function f is bounded if, and only if, it is bounded from above and from below. So

$$(f \text{ is bounded}) \Leftrightarrow (\exists M \in \mathbb{R}_+^*; \forall x \in D: |f(x)| \leq M).$$

Remark 4.1

If the function f is bounded on D, then the part $f(D)$ is bounded on $\mathbb{R}$. It accepts an upper bound and a lower bound, which we denote by $Sup_D\, f$ and $Inf_D\, f$ respectively.

Definition 4.3 Let $f: D \to \mathbb{R}$ be a function.

We say that f is increasing over D (strictly increasing, respectively) if and only if

$\forall x; y \in D: x < y \Longrightarrow f(x) \leq f(y)$ ($\forall x; y \in D: x < y \Longrightarrow f(x) < f(y)$, respectively).

We say that f is decreasing over D (strictly decreasing, respectively) if and only if

$\forall x; y \in D: x < y \Longrightarrow f(x) \geq f(y)$ ($\forall x; y \in D: x < y \Longrightarrow f(x) > f(y)$, respectively).

We say that f is constant over D if and only if $\forall x; y \in D: x \neq y \Longrightarrow f(x) = f(y)$.

Definition 4.4 Let $f: D \to \mathbb{R}$ be a function.

We say that f have a local maximum (local minimum, respectively) at point x_0 of D if:

$\exists \alpha \in \mathbb{R}_+^*; \forall x \in D: |x - x_0| < \alpha \Longrightarrow f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$, respectively).

And if $\forall x \in D: f(x) \le f(x_0)$ ($f(x) \ge f(x_0)$, respectively) we say that f have an absolute maximum (absolute minimum, respectively) at x_0.

4.2 limit of a function

4.2.1 Finite limit

Definition 4.5 (neighbourhood)

A subset of $\mathbb{R}$ is called the neighbourhood of a point $x_0 \in \mathbb{R}$ if it contain an open interval that include x_0. And we symbolize it by V_{x_0}.

Definition 4.6 (Finite limit)

Let f be a function, defined on a neighbourhood V_{x_0} of point x_0, with the possible exception of point x_0.

We say that the function f has a limit $\ell(\ell \in \mathbb{R})$ at point x_0 if, and only if,

$\forall \varepsilon > 0\,; \exists \delta > 0; \forall x \in V_{x_0}: 0 < |x - x_0| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon$, and we write $\lim\limits_{x \to x_0} f(x) = \ell$.

Remark

We say that f does not accept the number ℓ as a limit at x_0 if and only if

$$\exists \varepsilon > 0\,; \forall \delta > 0; \exists x \in V_{x_0}: 0 < |x - x_0| < \delta \text{ و } |f(x) - \ell| \ge \varepsilon.$$

proposition 4.1

If $\lim\limits_{x \to x_0} f(x) = \ell \neq 0$, then there exists a domain of the form$]x_0 - \alpha, x_0[\, \cup \,]x_0, x_0 + \alpha[$, with $\alpha > 0$, such that $f(x)$ has the same sign as ℓ.

Proof

For $\varepsilon = |\ell|$, then $\exists \alpha > 0; \forall x \in V_{x_0}: 0 < |x - x_0| < \alpha \Rightarrow |f(x) - \ell| < |\ell|$ from him

$$x \in \,]x_0 - \alpha, x_0[\, \cup \,]x_0, x_0 + \alpha[\Rightarrow \begin{cases} 2\ell < f(x) < 0\,;\ \ell < 0 \\ 0 < f(x) < 2\ell\,;\ \ell > 0 \end{cases}$$

$$\Rightarrow f(x) \text{ has the same sign as } \ell.$$

Examples

1) Let $f: x \to 5x - 7$ Be a function , using the definition prove that: $\lim\limits_{x \to 2} f(x) = 3$.

Since f is defined on $\mathbb{R}$, we can take $V_2 = \mathbb{R}$.(V_2 is a neighbourhood of point 2)

Let $\varepsilon \in \mathbb{R}_+^*$, we have $\forall x \in \mathbb{R}$:

$$|f(x) - 3| < \varepsilon \Leftrightarrow |5x - 7 - 3| < \varepsilon$$

$$\Leftrightarrow \ |x-2| < \frac{\varepsilon}{5}$$

So it is enough to take $\delta = \frac{\varepsilon}{5}$ to achieve the following:

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in \mathbb{R} : 0 < |x-2| < \delta \Longrightarrow |f(x) - 3| < \varepsilon.$$

2) Let $f: x \to x \to \frac{1}{x+1}$ Be a function , using the definition prove that: $\lim_{x \to 1} f(x) = \frac{1}{2}$..

Since f is defined on $\mathbb{R} - \{1\}$, we can take $V_1 = [0; +\infty[$ (V_1 is a neighbourhood of point 2)

Let $\varepsilon \in \mathbb{R}_+^*$, we have

$$\forall x \in V_1: \left|f(x) - \frac{1}{2}\right| = \left|\frac{1}{x+1} - \frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} < \frac{|x-1|}{2}.$$

Therefore, it suffices to take $\frac{|x-1|}{2} < \varepsilon$ to be $\left|f(x) - \frac{1}{2}\right| < \varepsilon$, from which

$\left|\frac{x-1}{2}\right| < \varepsilon \Leftrightarrow |x-1| < 2\varepsilon$. So it is enough to take $\delta = 2\varepsilon$ to achieve the following:

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_1: \ \ 0 < |x-1| < \delta \Longrightarrow \left|f(x) - \frac{1}{2}\right| < \varepsilon.$$

Definition 4 6

Let f be a function defined in the interval $V_{x_0} =]x_0, \mathrm{b}[$, we say that f have the limit ℓ from the right at x_0 if and only if

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < x - x_0 < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

we write $\lim_{\substack{> \\ x \to x_0}} f(x) = \ell$ or $\lim_{x \to x_0^+} f(x) = \ell$.

Let f be a function defined in the interval $V_{x_0} =]a, x_0[$, we say that f have the limit ℓ from the left at x_0 if and only if

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \ -\delta < x - x_0 < 0 \Longrightarrow |f(x) - \ell| < \varepsilon.$$

we write $\lim_{\substack{< \\ x \to x_0}} f(x) = \ell$ or $\lim_{x \to x_0^-} f(x) = \ell$.

Proposition 4.2

The limit at a point of a function exists if and only if the left limit and the right limit exist and are equal.

Example

Let the function f defined on $\mathbb{R}$ by $f(x) = \begin{cases} 3x - 1 & if \quad x \leq 1 \\ \frac{6}{x+2} & if \quad x > 1 \end{cases}$.

Prove that: $\lim\limits_{\substack{> \\ x\to 1}} f(x) = 2$ and $\lim\limits_{\substack{< \\ x\to 1}} f(x) = 2$ what do you conclude.

1) Let $V_1 =]-\infty; 1]$ and $\varepsilon \in \mathbb{R}_+^*$, we have

$$\forall x \in V_1:\ |f(x) - 2| < \varepsilon \Leftrightarrow |3x - 3| < \varepsilon$$

$$|3x - 3| < \varepsilon \Leftrightarrow 0 < |x - 1| < \frac{\varepsilon}{3}$$

$$\Leftrightarrow 0 < -x + 1 < \frac{\varepsilon}{3}$$

$$\Leftrightarrow -\frac{\varepsilon}{3} < x - 1 < 0$$

It is enough to take $\delta = \frac{\varepsilon}{3}$ to achieve the following:

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_1:\ 0 < 1 - x < \delta \Longrightarrow |f(x) - 2| < \varepsilon$$

Let $V_1 = [1; +\infty[$ and $\varepsilon \in \mathbb{R}_+^*$, we have

$$\forall x \in V_1:\ |f(x) - 2| = \frac{2|x - 1|}{x + 2} < \frac{2}{3}|x - 1|$$

So

$$\frac{2}{3}|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \frac{3}{2}\varepsilon \Leftrightarrow 0 < x - 1 < \frac{3}{2}\varepsilon$$

It is enough to take $\delta = \frac{3\varepsilon}{2}$ to achieve the following:

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_1:\ 0 < x - 1 < \delta \Longrightarrow |f(x) - 2| < \varepsilon$$

Conclusion: Since $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 2$ f accepts a limit at 1, which is 2.

Theorem 4.1

If a function f accepts a limit at x_0, then this limit is unique.

Proof

Let f accept two different limits ℓ and ℓ' where $\ell > \ell'$.

for $\varepsilon = \frac{\ell - \ell'}{2}$; $\exists \delta_1, \delta_2 > 0; \forall x \in V_{x_0}$:

$$0 < |x - x_0| < \delta_1 \Longrightarrow\ |f(x) - \ell| < \varepsilon = \frac{\ell - \ell'}{2}$$

and

$$0 < |x - x_0| < \delta_2 \Longrightarrow\ |f(x) - \ell'| < \varepsilon = \frac{\ell - \ell'}{2}$$

For $\delta = \min\{\delta_1, \delta_2\}$ Then $\forall x \in \mathrm{V}_{x_0}$:

$$0 < |x - x_0| < \delta \Longrightarrow |\ell - \ell'| = |f(x) - \ell - (f(x) - \ell')|$$
$$\Longrightarrow |\ell - \ell'| < \varepsilon + \varepsilon = 2\varepsilon$$
$$\Longrightarrow |\ell - \ell'| < |\ell - \ell'|$$

This is a contradiction. So $\ell = \ell'$

4.2.2 Limit of a function using sequences

Theorem 4.2

Let $f: D \to \mathbb{R}$ be a function and $x_0 \in D$. The following two conditions are equivalent.

1) $\lim\limits_{x \to x_0} f(x) = \ell$.

2) For all sequence (x_n) where $\forall n \in \mathbb{N}: x_n \in D \ \wedge \ x_n \neq x_0$ then:

$$(\lim_{n \to +\infty} x_n = x_0) \Longrightarrow (\lim_{n \to +\infty} f(x_n) = \ell)$$

Proof

Necessary condition

We impose $\lim\limits_{x \to x_0} f(x) = \ell$ and let (x_n) sequence where $\forall n \in \mathbb{N}: x_n \in D \ \wedge \ x_n \neq x_0$ and $\lim\limits_{\mathrm{n} \to \infty} x_n = x_0$.Let us prove that: $\lim\limits_{n \to +\infty} f(x_n) = \ell$.

For $\varepsilon > 0$ then $\exists \delta > 0; \forall x \in \mathrm{V}_{x_0}: 0 < |x - x_0| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon$. So

$\exists \mathrm{N} \in \mathbb{N};\ \forall n \in \mathbb{N}: n > \mathrm{N} \Longrightarrow |x_n - x_0| < \delta \Longrightarrow |f(x_n) - \ell| < \varepsilon$.

So $\forall \varepsilon > 0; \exists \mathrm{N} \in \mathbb{N};\ \forall n \in \mathbb{N}: n > \mathrm{N} \Longrightarrow |f(x_n) - \ell| < \varepsilon$.So $\lim\limits_{n \to +\infty} f(x_n) = \ell$.

Sufficient condition

We now assume that for every sequence (x_n) where $\forall n \in \mathbb{N}: x_n \in D \ \wedge \ \mathrm{x}_n \neq x_0$ then $(\lim\limits_{n \to +\infty} x_n = x_0) \Longrightarrow (\lim\limits_{n \to +\infty} f(x_n) = \ell)$.

Let us prove by contradiction that $\lim\limits_{x \to x_0} f(x) = \ell$.

Assume that $\lim\limits_{x \to x_0} f(x) \neq \ell$, that is $\exists \varepsilon > 0; \forall \delta > 0; \exists x \in \mathrm{V}_{x_0}: 0 < |x - x_0| < \delta$ and

$|f(x) - \ell| \geq \varepsilon$.

For $\delta = \frac{1}{n}$ then $\forall n \in \mathbb{N}^*; \exists\ x_n \neq x_0$ and $x_n \in \mathrm{V}_{x_0}$: $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - \ell| \geq \varepsilon$.

So $\lim\limits_{n \to +\infty} x_n = x_0$ and $\lim\limits_{n \to +\infty} f(x_n) \neq \ell$ (this is a contradiction).

Remark

To prove that a function f has no limit at x_0, it is enough to find two sequences (x_n) and (x'_n) that converge towards x_0 but $\lim_{n\to\infty} f(x'_n) \neq \lim_{n\to\infty} f(x_n)$ Or we are looking for a sequence (x_n) that converges toward x_0 but the sequence $\left(f(x_n)\right)_{n\in\mathbb{N}}$ diverges.

Example

Prove that the function $f: x \to \cos\frac{1}{x}$ does not accept a limit at 0.

Let the sequences (x_n) and (x'_n) where $\forall n \in \mathbb{N}^*: x_n = \frac{1}{2\pi n+\frac{\pi}{2}}$ ‘ $x'_n = \frac{1}{2\pi n+\pi}$. So

$\forall n \in \mathbb{N}^*: f(x'_n) = -1$; $f(x_n) = 0$. We have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = 0$ and $\lim f(x'_n) = -1 \neq \lim f(x_n) = 0$. So f does not accept a limit at 0.

4.2.3 Infinite limits

We say a subset of $\mathbb{R}$ is a neighbourhood of $+\infty$ ($-\infty$, respectively) if it contains an open interval of the form $]a, +\infty[$ ($]-\infty, b[$, respectively) And we symbolize it with $V_{+\infty}$ ($V_{-\infty}$, respectively).

Definitions

$(\forall\varepsilon > 0; \exists A > 0; \forall x \in V_{+\infty}: x > A \Longrightarrow |f(x) - \ell| < \varepsilon) \Leftrightarrow (\lim_{x\to+\infty} f(x) = \ell\,)$

$(\forall\varepsilon > 0; \exists A > 0; \forall x \in V_{-\infty}: x < -A \Longrightarrow |f(x) - \ell| < \varepsilon) \Leftrightarrow (\lim_{x\to-\infty} f(x) = \ell)$

$\left(\forall A > 0; \exists\delta > 0; \forall x \in V_{x_0}: |x - x_0| < \delta \Longrightarrow f(x) > A\right) \Leftrightarrow (\lim_{x\to x_0} f(x) = +\infty)$

$\left(\forall A > 0; \exists\delta > 0; \forall x \in V_{x_0}: |x - x_0| < \delta \Longrightarrow f(x) < -A\right) \Leftrightarrow (\lim_{x\to x_0} f(x) = -\infty)$

$(\forall A > 0; \exists B > 0; \forall x \in V_{+\infty}: x > B \Longrightarrow f(x) > A) \Leftrightarrow (\lim_{x\to+\infty} f(x) = +\infty)$

$(\forall A > 0; \exists B > 0; \forall x \in V_{+\infty}: x > B \Longrightarrow f(x) < -A) \Leftrightarrow (\lim_{x\to+\infty} f(x) = -\infty)$

$(\forall A > 0; \exists B > 0; \forall x \in V_{-\infty}: x < -B \Longrightarrow f(x) > A) \Leftrightarrow (\lim_{x\to-\infty} f(x) = +\infty)$

$(\forall A > 0; \exists B > 0; \forall x \in V_{-\infty}: x < -B \Longrightarrow f(x) < -A) \Leftrightarrow (\lim_{x\to-\infty} f(x) = -\infty)$

Examples

1) Prove that $\lim_{x\to\infty} \frac{2x}{x-1} = 2$.

The function $x \to \frac{2x}{x-1}$ is defined on $V_{+\infty} =]1; +\infty[$, for $\varepsilon \in \mathbb{R}_+^*$ we have

$$\forall x \in V_{+\infty}: |f(x) - 2| < \varepsilon \Leftrightarrow \frac{2}{|x-1|} < \varepsilon \Leftrightarrow \frac{2}{x-1} < \varepsilon \Leftrightarrow x > \frac{2}{\varepsilon} + 1$$

Therefore, it is sufficient to choose $B = \frac{2}{\varepsilon} + 1$ to obtain:

$$\forall \varepsilon > 0;\ \exists B \in \mathbb{R}_+^*;\ \forall x \in V_{+\infty}: x > B \Longrightarrow |f(x) - 2| < \varepsilon$$

2) Prove that $\lim\limits_{\substack{x \to 1 \\ <}} \frac{2x}{x-1} = -\infty$.

Let $V_1 =]0;1[$, for $A \in \mathbb{R}_+^*$ we have

$$\forall x \in V_1: f(x) < -A \Leftrightarrow \frac{2x}{x-1} < -A \Leftrightarrow 2 + \frac{2}{x-1} < -A$$

$$\Leftrightarrow 0 > x - 1 > \frac{2}{-A-2}$$

$$\Leftrightarrow -\frac{2}{A+2} < x - 1 < 0$$

Therefore, it is sufficient to choose $\delta = \frac{2}{A+2}$ to obtain:

$\forall A > 0;\ \exists \delta \in \mathbb{R}_+^*;\ \forall x \in V_1: 0 < 1 - x < \delta \Longrightarrow f(x) < -A$.

4.2.4 Operation on limits

Theorem 4.3

Let f and g be functions defined on the neighbourhood V_{x_0}, with the possible exception of x_0, where

$$\forall x \in V_{x_0}: f(x) < g(x)\ (\text{or } f(x) \leq g(x)\ )$$

1) If $\lim\limits_{x \to x_0} f(x) = \ell$ and $\lim\limits_{x \to x_0} g(x) = \ell'$ then $\ell \leq \ell'$.

2)) If $\lim\limits_{x \to x_0} f(x) = +\infty$ then $\lim\limits_{x \to x_0} g(x) = +\infty$.

3) $\lim\limits_{x \to x_0} g(x) = -\infty$ then $\lim\limits_{x \to x_0} f(x) = -\infty$.

Let f,g and h be functions defined on the neighbourhood V_{x_0}, with the possible exception of x_0, where $\forall x \in V_{x_0}: h(x) < f(x) < g(x)$ (or $h(x) \leq f(x) \leq g(x)$) and $\lim\limits_{x \to x_0} g(x) = \lim\limits_{x \to x_0} h(x) = \ell$, then $\lim\limits_{x \to x_0} f(x) = \ell$.

Proof

Assume that $\forall x \in V_{x_0}: f(x) < g(x)$ and $\lim\limits_{x \to x_0} f(x) = \ell$, $\lim\limits_{x \to x_0} g(x) = \ell'$ and suppose that

$\ell > \ell'$. For $\varepsilon = \frac{\ell - \ell'}{2}$ then

$$\exists\delta_1 > 0: 0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - \ell| < \varepsilon \Rightarrow \frac{\ell + \ell'}{2} < f(x) < \frac{3\ell - \ell'}{2}$$

$$\exists\delta_2 > 0: 0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - \ell'| < \varepsilon \Rightarrow \frac{3\ell' - \ell}{2} < g(x) < \frac{\ell + \ell'}{2}$$

Bu taking $\delta = \min\{\delta_1, \delta_2\}$ then $0 < |x - x_0| < \delta \Rightarrow g(x) < \frac{\ell+\ell'}{2} < f(x)$ this is contradiction the hypothesis .$\forall x \in V_{x_0}: f(x) < g(x)$.

Theorem 4.4

If f and g are functions defined in the neighbourhood V_{x_0}, with the possible exception of x_0, and have the limits ℓ, ℓ', at x_0 respectively, then the functions $f+\ g$, $f\ g$, λf, $|f|$ it has the limits $\ell + \ell'$, $\lambda\ell$, $\ell\ell'$, $|\ell|$, at x_0 respectively. And if $\ell' \neq 0$, then the function $\frac{1}{g}$ it has the limit $\frac{1}{\ell'}$ at x_0.

Proof (Let us prove the last case)

Assume that $\lim_{x\to x_0} g(x) = \ell' \neq 0$ for $\varepsilon = \frac{|\ell'|}{2}$, then

$$\exists\delta_1 > 0: 0 < |x - x_0| < \delta_1 \Rightarrow |g(x) - \ell'| < \frac{|\ell'|}{2}$$

$$\Rightarrow ||g(x)| - |\ell'|| < \frac{|\ell'|}{2}$$

$$\Rightarrow \frac{|\ell'|}{2} < |g(x)| < \frac{3|\ell'|}{2}$$

$$\Rightarrow \frac{1}{|g(x)|} < \frac{2}{|\ell'|}.$$

On the other hand we have:

$$\forall\varepsilon > 0\,; \exists\delta_2 > 0; \forall x \in V_{x_0}: 0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - \ell'| < \varepsilon.$$

For $\delta = \min\{\delta_1, \delta_2\}$, then

.$0 < |x - x_0| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{\ell'}\right| = \left|\frac{\ell' - g(x)}{\ell' g(x)}\right| < \frac{2|g(x) - \ell'|}{|\ell'|^2} < \frac{2\varepsilon}{|\ell'|^2} = \varepsilon'$

4.2.5 Indeterminate form

We say that we are in the presence of an indeterminate form. If when $x \to x_0$

1) $f \to +\infty$ and $g \to -\infty$ then $f + g \to$ indeterminate form $+\infty - \infty$.

2) $f \to \infty$ and $g \to 0$ then $f.g \to$ indeterminate form $\infty.0$.

3) $f \to \infty$ and $g \to \infty$ then $\frac{f}{g} \to$ indeterminate form $\frac{\infty}{\infty}$.

4) $f \to 0$ and $g \to 0$ then $\frac{f}{g} \to$ indeterminate form $\frac{0}{0}$.

5) $f \to 0$ and $g \to 0$ then $f^g \to$ indeterminate form 0^0.

6) $f \to \infty$ and $g \to 0$ then $f^g \to$ indeterminate form ∞^0.

7) $f \to 1$ and $g \to \infty$ then $f^g \to$ indeterminate form 1^∞.

Remarks

1) The indeterminate forms $\infty.0$, $\frac{\infty}{\infty}$ can be reduced to the form $\frac{0}{0}$. by writing $\frac{f}{g} = \frac{\frac{1}{g}}{\frac{1}{f}}$ in (3) and $f.g = \frac{g}{\frac{1}{f}}$ in (2)/

2) The indeterminate forms 0^0, ∞^0, 1^∞ can be reduced to the form $\infty.0$ by passing the logarithm.

Exercise

1) Calculate the limits: $\lim_{x\to -1} \frac{x^2+3x+2}{x^4+1}$.

2) Using the limit $\lim_{h\to 0} \frac{\ln(h+1)}{h}$, calculate the limits:a) $\lim_{x\to\infty} x\ln\frac{x+1}{x-2}$, b) $\lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^x$.

Solution

1) $\lim_{x\to -1} \frac{x^2+3x+2}{x^4+1} = \text{IF } \frac{0}{0}$. So

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^4+1} = \lim_{x\to -1} \frac{(x+2)(x+1)}{(x^3-x^2+x-1)(x+1)} = \lim_{x\to -1} \frac{(x+2)}{(x^3-x^2+x-1)} = -\frac{1}{4}.$$

2) a) $\lim_{x\to\infty} x\ln\frac{x+1}{x-2} = \text{IF } \infty.0$. So

Putting $\frac{x+1}{x-2} = 1 + h$ we get $h = \frac{-3}{x-2}$ and for $x \to \infty$ then $h \to 0$ therefore

$$\lim_{x\to\infty} x\ln\frac{x+1}{x-2} = \lim_{\substack{x\to\infty \\ h\to 0}} xh\frac{\ln(1+h)}{h} = \lim_{\substack{x\to\infty \\ h\to 0}} \frac{-3x}{x-2}\frac{\ln(1+h)}{h} = -3 \times 1 = -3.$$

b) $\lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^x = \text{IF } 1^\infty$. So

Putting $f(x) = \left(\frac{x+1}{x-2}\right)^x$ and passing the logarithm we get $g(x) = \ln f(x) = x\ln\frac{x+1}{x-2}$, according to the first question we have $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x\ln\frac{x+1}{x-2} = -3$. So $\ln f(x) = -3$ and we obtain $\lim_{x\to\infty} f(x) = e^{-3}$.

4.2.6 Cauchy's criterion for functions:

Theorem 4.4

A function f has a finite limit at x_0 if and only if

$\forall \varepsilon > 0 ; \exists \delta > 0; \forall x', x'' \in V_{x_0}: (0 < |x' - x_0| < \delta \text{ and } 0 < |x'' - x_0| < \delta) \Longrightarrow$

$$|f(x') - f(x'')| < \varepsilon$$

Proof

Necessary condition

Assume that $\lim_{x \to x_0} f(x) = \ell$, then

$\forall \varepsilon > 0 ; \exists \delta > 0; \forall x', x'' \in V_{x_0}: (0 < |x' - x_0| < \delta \text{ and } 0 < |x'' - x_0| < \delta) \Longrightarrow$

$$|f(x') - \ell| < \frac{\varepsilon}{2} \text{ و } |f(x'') - \ell| < \frac{\varepsilon}{2}.$$

So

$$|f(x') - f(x'')| = |f(x') - \ell - (f(x'') - \ell)| \leq |f(x') - \ell| + |(f(x'') - \ell)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Sufficient condition

Assume that $\forall \varepsilon > 0 ; \exists \delta > 0; \forall x', x'' \in V_{x_0}$:

$$\left(0 < |x' - x_0| < \delta \text{ و } 0 < |x'' - x_0| < \delta\right) \Longrightarrow |f(x') - \ell| < \frac{\varepsilon}{2} \text{ و } |f(x'') - \ell| < \frac{\varepsilon}{2}.$$

Let (x_n) be a sequence of V_{x_0} elements where $\forall n \in \mathbb{N}: x_n \neq x_0$ and $\lim_{n \to \infty} x_n = x_0$.

So for $\delta > 0$, then $\exists N_0 \in \mathbb{N}: \forall n \in \mathbb{N}; n > N_0 \Longrightarrow |x_n - x_0| < \delta$.

So $\forall p, q \in \mathbb{N}: p > N_0 \text{ and } q > N_0 \Longrightarrow 0 < |x_p - x_0| < \delta \;\; and \; 0 < |x_q - x_0| < \delta$

$$\Longrightarrow |f(x_p) - f(x_q)| < \varepsilon.$$

So (x_n) is a Cauchy sequence, and therefore convergent.

Let us now show that the limit $\lim_{n \to \infty} f(x_n)$ is independent of the choice of sequence (x_n).

Let (x_n) and (x'_n) where $\lim_{n \to \infty} x'_n = \lim_{n \to \infty} x_n = x_0$.

So $\exists N \in \mathbb{N}; \; \forall n \in \mathbb{N}: n > N \Longrightarrow \; (0 < |x_n - x_0| < \delta \text{ and } 0 < |x'_n - x_0| < \delta)$

$$\Longrightarrow |f(x_n) - f(x'_n)| < \varepsilon.$$

So

$$\lim_{n \to \infty}\left(f(x_n) - f(x'_n)\right) = 0,$$

we obtain

$\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(x'_n)$.

4.2.7 Comparison of functions in the neighbourhood of a point - Landau notation

Let f and g be a functions defined in the neighbourhood V_{x_0} of the point x_0, with the possible exception of x_0

Definition 4.8

We say that f is negligible in front of g when $x \longrightarrow x_0$, and we write $f = o(g)$, if

$$\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < |x - x_0| < \delta \Longrightarrow |f(x)| \leq \varepsilon |g(x)|.$$

Definition 4.9

We say that f is dominated by g when $x \longrightarrow x_0$, and we write $f = o(g)$, if

$$\exists k > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < |x - x_0| < \delta \Longrightarrow |f(x)| \leq k |g(x)|.$$

The symbols o and O are called Landau symbols.

Corollary 4.1

If g is non-zero on $V_{x_0} - \{x_0\}$ then:

$f = o(g) \Leftrightarrow \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = 0.$

$f = O(g) \Leftrightarrow \left|\frac{f(x)}{g(x)}\right|$ is bounded in V_{x_0}.

And if $g = 1$, then

$f = o(1) \Leftrightarrow \lim\limits_{x\to x_0} f(x) = 0$ and $f = O(1) \Leftrightarrow f$ is bounded in V_{x_0}.

Remark

We obtain a similar definition for $x_0 = +\infty$ and $x_0 = -\infty$.

Examples

1) When $x \longrightarrow 0$ we have.

$x^3 = o(x^2)$, $x^2 \cos\frac{1}{x} = O(x^2)$, $\left(\frac{1}{x}\right)^3 = o\left(\left(\frac{1}{x}\right)^4\right)$.

2) When $x \longrightarrow +\infty$ we have

$x^2 = o(x^3)$, $x^2 \sin x = O(x^2)$, $\left(\frac{1}{x}\right)^4 = o\left(\left(\frac{1}{x}\right)^3\right)$.

Theorem 4.5

1) $f = gh \Leftrightarrow f = o(g)$ where $h = o(1)$.

2) $f = gh \Leftrightarrow f = O(g)$ where $h = O(1)$.

Proof (Let's prove 1)

Necessary condition

Assume that $f = o(g)$.

We put $h(x) = \begin{cases} \frac{f(x)}{g(x)}, & g(x) \neq 0 \\ 0 & , g(x) = 0 \end{cases}$.

We have $f = o(g) \Leftrightarrow \forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < |x - x_0| < \delta \Longrightarrow |f(x)| \leq \varepsilon |g(x)|$.

First: Let us prove that $f = gh$.

If $g(x) = 0$ then $0 < |x - x_0| < \delta \Longrightarrow |f(x)| \leq \varepsilon |g(x)| = 0$, we get $f = gh$.

If $g(x) \neq 0$ then $f(x) = g(x) \frac{f(x)}{g(x)}$, we get $f = gh$.

second:

Let us show that $h = o(1)$, i:e $\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < |x - x_0| < \delta \Longrightarrow |h(x)| \leq \varepsilon$

If $g(x) = 0$ then $h(x)$=0, i.e $|h(x)| \leq \varepsilon$

If $g(x) \neq 0$ then $|f(x)| \leq \varepsilon |g(x)|$ and from it $\left|\frac{f(x)}{g(x)}\right| \leq \varepsilon$ i.e $|h(x)| \leq \varepsilon$.

Sufficient condition

Assume that $f = gh$ and $h = o(1)$ and show that $f = o(g)$.

We have $\big(h = o(1)\big) \Leftrightarrow \big(\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: \quad 0 < |x - x_0| < \delta \Longrightarrow |h(x)| \leq \varepsilon\big)$ and from there $|f(x)| = |h(x)g(x)| \leq \varepsilon |g(x)|$ i.e. $f = o(g)$.

In the same way we prove property 2.

Note: The previous two properties are summarized in the following writing.

$$o(g) = g.o(1) \quad \text{and} \quad O(g) = g.O(1)$$

Properties

1) $f = O(g)$ and $h = O(g) \Longrightarrow f + h = O(g)$.

2) $f = o(g)$ and $h = o(g) \Longrightarrow f + h = o(g)$.

3) $f = o(g)$ and $h = O(1) \Longrightarrow fh = o(g)$.

4) $f = o(g)$ and $h = O(g) \Longrightarrow f + h = O(g)$.

5) $f = O(g)$ and $h = O(1) \Longrightarrow fh = O(g)$.

6) $h = O(f)$ and $f = o(g) \Longrightarrow h = o(g)$.

7) $h = o(f)$ and $f = O(g) \Longrightarrow h = o(g)$.

Note

The previous properties are summarized in the following writing.

1) $O(g) + O(g) = O(g)$.

2) $o(g) + o(g) = o(g)$.

3) $o(g)O(1) = o(g)$.

4) $o(g) + O(g) = O(g)$.

5) $O(g).O(1) = O(g)$.

6) $O\big(o(g)\big) = o(g)$.

7) $o\big(O(g)\big) = o(g)$.

4.2.8 Equivalent functions:

Let f and g be a functions defined in the neighbourhood V_{x_0} of the point x_0, with the possible exception of x_0.

Definition 4.11

We say that f is equivalent to g for $x \longrightarrow x_0$ and write $f \sim g$ if $f - g = o(f)$ for $x \longrightarrow x_0$.

Results 4.1

1) $f - g = o(f) \Leftrightarrow f - g = o(g)$.

2) The relation $\sim$ is an equivalence relation on the set of functions defined in the neighborhood $\mathrm{V}_{x_0} - \{x_0\}$ of the point x_0.

3) If f and g are non-zero on $\mathrm{V}_{x_0} - \{x_0\}$ then: $f \sim g \Leftrightarrow \lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

Theorem 4.7

Let f , g , f_1 and g_1 be a functions defined in the neighbourhood V_{x_0} of the point x_0, with the possible exception of x_0 where $f \sim f_1$ and $g \sim g_1$ for $x \longrightarrow x_0$. If

If the limit $\lim\limits_{x \to x_0} \frac{f(x)}{(x)}$ it exists then the limit $\lim\limits_{x \to x_0} \frac{f_1(x)}{g_1(x)}$ olso exists and we have:

$$\lim_{x \to x_0} \frac{f_1(x)}{g_1(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

Proof

Since $\frac{f(x)}{g(x)}$ accepts a limit when $x \to x_0$, there is a neighbourhood V_{x_0} to the point x_0, such that g is non-zero on $V_{x_0} - \{x_0\}$ and that $g \sim g_1$ (that is, $|g(x)| \leq \varepsilon|g_1(x)|$) then g_1 is also non-zero on $V_{x_0} - \{x_0\}$ and hence

$$\begin{cases} f \sim f_1 \\ g \sim g_1 \end{cases} \Rightarrow \begin{cases} f_1 \sim f \\ g_1 \sim g \end{cases} \Rightarrow \begin{cases} f_1 = f\big(1 + o(1)\big) \\ g_1 = g\big(1 + o(1)\big) \end{cases} \Rightarrow \frac{f_1}{g_1} = \frac{f}{g}\frac{\big(1 + o(1)\big)}{\big(1 + o(1)\big)}.$$

And since $\frac{(1+o(1))}{(1+o(1))} = 1 + o(1) \longrightarrow 1$, then $\lim_{x \to x_0} \frac{f_1(x)}{g_1(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$.

Remark

Note: The concept of equivalent functions is used in calculating limits, especially in removing indeterminacy.

Examples

1) Calculate the limit $\lim_{x \to 0} \frac{\sqrt{4+x}-2}{\sqrt[3]{x+1}-1}$.

For $x \to 0$ we have $\sqrt{4+}- 2 \sim \frac{1}{2}x$ and $\sqrt[3]{x+1} - 1 \sim \frac{1}{3}x$, and from it

$$\lim_{x \to 0} \frac{\sqrt{4+x}-2}{\sqrt[3]{x+1}-1} = \lim_{x \to 0} \frac{\frac{1}{2}x}{\frac{1}{3}x} = \frac{3}{2}.$$

2) Calculate the limit $\lim_{x \to +\infty} \frac{\sqrt{x^2-2x}+x}{2+xe^{\frac{1}{x}}}$.

For $x \to +\infty$ we have $\sqrt{x^2 - 2x} + x \sim 2x$ and $2 + xe^{\frac{1}{x}} \sim x$, and from it

$$\lim_{x \to +\infty} \frac{\sqrt{x^2-2x}+x}{2+xe^{\frac{1}{x}}} = \lim_{x \to +\infty} \frac{2x}{x} = 2.$$

4.3 Continuous functions:

Definitions 4.12

1) Let f be a function defined on the neighbourhood V_{x_0} of the point x_0. We say that f is continuous at

x_0 if and only if: $\lim_{x \to x_0} f(x) = f(x_0)$.

In other words f is continuous at x_0 if and only if:

$$\big(\forall \varepsilon > 0; \exists \delta > 0; \forall x \in V_{x_0}: 0 < |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon\big).$$

2) Let f be a function defined on the neighbourhood V_{x_0} of the form $[x_0, b[$. We say that f is continuous at x_0 from the right if and only if: $\lim\limits_{\substack{> \\ x \to x_0}} f(x) = f(x_0)$.

3) Let f be a function defined on the neighbourhood V_{x_0} of the form $]a, x_0]$. We say that f is continuous at x_0 from the left if and only: $\lim\limits_{\substack{< \\ x \to x_0}} f(x) = f(x_0)$.

Result 4.2

A function f is continuous at x_0 if and only if it is continuous at x_0 from the right and from the left

Examples

1) Let the function f defined on $\mathbb{R}$ by $f(x) = \begin{cases} \frac{|x^2-1|}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } \quad x = 1 \end{cases}$.

$\lim\limits_{\substack{> \\ x \to 1}} f(x) = 2 = f(1) \implies f$ is continuous at $x_0 = 1$, from the right.

$\lim\limits_{\substack{< \\ x \to 1}} f(x) = -2 \neq f(1) \implies f$ is discontinuous at $x_0 = 1$, from the left. So f is discontinuous at $x_0 = 1$.

Definition 4.13

Le I be a interval of $\mathbb{R}$.

We say that a function f is continuous on the interval I if and only if it is continuous at every point in this interval. We denote the set of continuous functions on the interval I by $C(I)$.

We say that the function f is continuous uniformly over the domain I if and only if

$$\forall \varepsilon > 0; \exists \delta > 0: \forall x', x" \in I: |x' - x"| < \delta \implies |f(x') - f(x")| < \varepsilon.$$

It is clear from the definition that every uniformly continuous function in the interval I is continuous in this interval (the opposite is not always true).

4.3.1 Continuous functions in a closed interval

Theorem 4.8

Every continuous function in a closed interval $[a, b]$ is uniformly continuous in this interval.

Proof

We assume that f is continuous and uniformly discontinuous on $[a, b]$ i.e.

$$\exists \varepsilon > 0; \forall \delta > 0: \exists x', x" \in [a, b]: |x' - x"| < \delta \text{ and } |f(x') - f(x")| \geq \varepsilon.$$

We put $\delta = \frac{1}{n} > 0$ where $n \in \mathbb{N}^*$ and from it:

$$\exists \varepsilon > 0; \forall n \in \mathbb{N}^*; \exists x'_n, x''_n \in [a,b]: |x'_n - x''_n| < \frac{1}{n} \text{ and } |f(x'_n) - f(x''_n)| \geq \varepsilon.$$

Since the sequence (x'_n) is bounded, according to the BOLZANO-WEIERSTRASS theorem, then a subsequence (x'_{n_k}) can be extracted from it that converges towards $\bar{x}$ in $[a,b]$ and since $\forall k \in \mathbb{N}: \left|x'_{n_k} - x''_{n_k}\right| < \frac{1}{n_k}$, the partial sequence (x''_{n_k}) also converges towards $\bar{x}$, and since f is continuous at $\bar{x}$, then $\lim\limits_{k\to\infty}\left(f(x'_{n_k}) - f(x''_{n_k})\right) = f(\bar{x}) - f(\bar{x}) = 0$. This is a contradiction because $\forall k \in \mathbb{N}: \left|f(x'_{n_k}) - f(x''_{n_k})\right| \geq \varepsilon$.

Theorem 4.9

Every continuous function on the closed interval $[a,b]$, is bounded.

Proof

Assume that f continuous and unbounded on the interval $[a,b]$, i.e. $\forall n \in \mathbb{N}; \exists x_n \in [a,b]: |f(x_n)| > n$.

Since the sequence (x_n) is bounded, it is possible to extract from it a partial sequence (x_{n_k}) that converges towards $\bar{x}$ from $[a,b]$. Since f is continuous at $\bar{x}$, then $\lim\limits_{k\to\infty}\left|f(x_{n_k})\right| = |f(\bar{x})|$.

This is a contradiction because $\forall k \in \mathbb{N}: |f(n_k)| > n_k \geq k$, and hence $\lim\limits_{k\to\infty}\left|f(x_{n_k})\right| = +\infty$.

Theorem 4.10

Let f be a continuous function on a closed interval $[a;b]$.

Then f attains its upper and lower bounds on $[a;b]$, i.e. there exist some points $c, d \in [a,b]$ such that $f(c) = \sup_{x\in[a\,;b]} f(x)$ and $f(d) = inf_{x\in[a\,;b]} f(x)$.

Proof

Let $M = \sup_{x\in[a\,;b]} f(x)$. And assume that $\forall x \in [a\,;b]: f(x) \neq M$ i.e. $\forall x \in [a;b]: f(x) \neq M$.

So the function g defined on $[a;b]$ by $\forall x \in [a;b]: g(x) = \frac{1}{M - f(x)}$ it is continuous and strictly positive and therefore it is bounded to this interval, i.e.: $\exists m > 0; \forall x \in [a;b]: g(x) \leq m$ or

$\exists m > 0; \forall x \in [a;b]: f(x) \leq M - \frac{1}{m}$. This contradicts the hypothesis $M = \sup_{x\in[a;b]} f(x)$.

Theorem 4.11

Let f be a continuous function in the interval $[a;b]$, if the signs of $f(a)$ and $f(b)$ are different, then there is at least a point c in the interval $]a;b[$ satisfies: $f(c) = 0$.

Proof

Assume that $f(a) < 0$ and $f(b) > 0$. Let the set $\mathrm{E} = \{x \in [a;b]/f(x) > 0\}$, then $\mathrm{E} \neq \emptyset$ because $b \in \mathrm{E}$. We put $\inf \mathrm{E} = c$ and let us prove that: $f(c) = 0$.

Assume that $f(c) \neq 0$ Since f is continuous at c, there exists at least a interval of the form $I =]c-\alpha; c+\alpha[\subset [a;b]$ with $\alpha > 0$, where $f(x)$ and $f(c)$ have the same sign. (See Proposition 1.3).So

if $f(c) > 0$, then $\forall x \in I: f(x) > 0$ by taking $x = c - \frac{\alpha}{2}$ we get $f\left(c - \frac{\alpha}{2}\right) > 0$ so $c - \frac{\alpha}{2} \in \mathrm{E}$ and therefore $c - \frac{\alpha}{2} \geq c = \inf \mathrm{E}$. and this is a contradiction.

if $f(c) < 0$, then $\forall x \in I: f(x) < 0$.

We have $\inf \mathrm{E} = c \Longrightarrow \exists x_0 \in \mathrm{E}: c + \alpha > x_0 \geq c \Longrightarrow x_0 \in I \Longrightarrow f(x_0) < 0$. This is a contradiction because $x_0 \in \mathrm{E} \Longrightarrow f(x_0) > 0$.So $f(c) = 0$.

Theorem 4.12

Let f be a continuous function in the interval $[a;b]$. For every real number λ between $f(a)$ and $f(b)$, there exists at least one real number c of the interval $[a;b]$ satisfies: $f(c) = \lambda$.

Proof

case 1: If $\lambda = f(a)$ it is enough to take $c = a$, but if $\lambda = f(b)$ it is enough to take $c = b$.

case 2: If $\lambda \neq f(a)$ and $\lambda \neq f(b)$. Then the function g defined on the interval $[a;b]$ by

$g(x) = f(x) - \lambda$, satisfies the conditions of Theorem 4.11, So there exists at least one real number c of the interval $[a;b]$ where $g(c) = 0$ and from which we get $f(c) = \lambda$.

Proposition 3.2

Let I be an interval of $\mathbb{R}$, and f a real function.

If the function f is continuous on I, then the image of the interval I by the function f is a interval of $\mathbb{R}$, i.e. the set $f(I)$ is a interval.

Proof

Let $y_1; y_2$ be two numbers of $f(I)$ where $y_1 \leq y_2$ then there are at least two numbers x_1, x_2 of the interval I where $y_1 = f(x_1)$ and $y_2 = f(x_2)$ according to the theorem 4.12, then for every number y where $y_1 \leq y \leq y_2$, there exists at least number x confined between x_1 and x_2 (i.e. $x \in I$), where $y = f(x)$ therefore $y \in f(I)$.

4.3.2 Extension by continuity

Definition 4 14

Let f be a function defined on the domain I. With exception of the point x_0 of I, we assume that $\lim_{x \to x_0} f(x) = \ell$. Then the function $\tilde{f}$, defined by $\tilde{f}(x) = \begin{cases} f(x) & ; x \in I - \{x_0\} \\ \ell & ; x = x_0 \end{cases}$, coincides with f on $I - \{x_0\}$ and is continuous at x_0. The function $\tilde{f}$ is called the extension of f with continuity at x_0.

Example

Let f be a function defined on $\mathbb{R}^*$ by $f(x) = \frac{\sin 2x}{x}$. Since $\lim_{x \to 0} \frac{\sin 2x}{x} = 2$, then f can be extended by continuity at $x_0 = 0$ to the function $\tilde{f}$ where: $\tilde{f}(x) = \begin{cases} \frac{\sin 2x}{x} & ; x \neq 0 \\ 2 & ; x \neq 0 \end{cases}$.

4.3.3 Properties of monotone functions on an interval

Theorem 4.13

Let $f:]a, b[\to \mathbb{R}$ be a monotonic function where $-\infty < a < b < +\infty$, then the limits $\lim_{\substack{x \to a \\ >}} f(x)$ ، $\lim_{\substack{x \to b \\ <}} f(x)$, are exists (finite or infinite) and we have

If f increasing $\Longrightarrow -\infty \leq \inf_{x \in]a,b[} f(x) = \lim_{\substack{x \to a \\ >}} f(x) \leq \lim_{\substack{x \to b \\ <}} f(x) = \sup_{x \in]a,b[} f(x) \leq +\infty$

If f decreasing $\Longrightarrow -\infty \leq \inf_{x \in]a,b[} f(x) = \lim_{\substack{x \to b \\ <}} f(x) \leq \lim_{\substack{x \to a \\ >}} f(x) = \sup_{x \in]a,b[} f(x) \leq +\infty$

Proof

Assume that f increasing and $\sup_{x \in]a,b[} f(x) = M < +\infty$ and let us prove that: $\lim_{\substack{x \to b \\ <}} f(x) = M$.

We have $\sup_{x \in]a,b[} f(x) = M \Longrightarrow \forall \varepsilon > 0; \exists \alpha \in]a, b[: M - \varepsilon < f(\alpha) \leq M$.

By putting $\delta = b - \alpha > 0$, then $b - \delta < x < b \Rightarrow \alpha < x < b \quad \overset{f \text{ increasing}}{\Longrightarrow} \quad f(\alpha) \leq f(x)$

$$\Rightarrow M - \varepsilon < f(\alpha) \leq f(x) \leq M < M + \varepsilon$$

$$\Rightarrow M - \varepsilon < f(x) < M + \varepsilon.$$

So $\forall \varepsilon > 0; \exists \delta > 0: -\delta < x - b < 0 \Rightarrow |f(x) - M| < \varepsilon$ we get $\lim_{\substack{x \to b \\ <}} f(x) = M$.

In the same way we prove the second case.

Corollary 4.1

1) Let $f:]a, b[\to \mathbb{R}$ be a monotonic function then:

a) If f increasing $\Longrightarrow f(a) \leq \lim_{\substack{x \to a \\ >}} f(x) \leq \lim_{\substack{x \to b \\ <}} f(x) \leq f(b)$.

b) If f decreasing $\Longrightarrow f(b) \leq \lim_{\substack{x \to b \\ <}} f(x) \leq \lim_{\substack{x \to a \\ >}} f(x) \leq f(a)$.

2) Let I be an interval of $\mathbb{R}$ bounded by a and b ($a < b$), and let $f: [a, b] \to \mathbb{R}$ be an increasing function. For each x_0, where $a < x_0 < b$ then:

a) $-\infty < f(x_0 - 0) \leq f(x_0) \leq f(x_0 + 0) < +\infty$.

b) If $a \in I \Longrightarrow f(a) \leq f(a + 0) < +\infty$.

c) If $b \in I \Longrightarrow -\infty < f(b - 0) \leq f(b)$.

Remark

We obtain a corollary similar to corollary 4.1 if f is decreasing over the interval I.

Theorem 4.14

Let I be an interval of $\mathbb{R}$ and let $f: [a, b] \to \mathbb{R}$ be an monotonic function Then f is continuous on I if and only if $f(I)$ is a interval.

Proof

Necessary conditions

According to Proposition 2.3, if f is continuous, then $f(I)$ is an interval.

sufficient condition

We assume f is increasing and $f(I)$ is a interval and prove that f is continuous on I.

Suppose the opposite and let x_0 be a point of discontinuity of f. As f is increasing, then at least one of the relations $f(x_0) < f(x_0 + 0)$, $f(x_0 - 0) < f(x_0)$. is verified (According to corollary 4.1).

Assume, for example, that $f(x_0) < f(x_0 + 0)$ in this case, then for each x of I, we have

$x \leq x_0 \Rightarrow f(x) < f(x_0)$ and $x > x_0 \Rightarrow f(x) \geq f(x_0 + 0)$ that is $](x_0), f(x_0 + 0)[\cap f(I) = \emptyset$.

Let $x_1 \in I$ where $x_1 > x_0$ then $f(x_0) \in f(I)$ and $f(x_1) \in f(I)$ and from it $[f(x_0), f(x_1)] \subset f(I)$ (because $f(I)$ is a interval) and since $f(x_1) > f(x_0 + 0)$ then $]f(x_0), f(x_0 + 0)[\subset [f(x_0), f(x_1)]$

i.e. $]f(x_0), f(x_0 + 0)[\cap f(I) \neq \emptyset$. This is a contradiction.

4.4.3 The inverse function of a strictly monotonic continuous function

Theorem 4.15

Let I be an interval of $\mathbb{R}$ and $f: I \to \mathbb{R}$ a real function.

If f is continuous and strictly monotonic over the interval I, then f is a bijective of the interval I to the interval $f(I)$. Therefore, f accepts an inverse function that we denote by f^{-1}, which is defined, continuous, and strictly monotonic over the interval $f(I)$ and has the same direction of change of f, and we have

$$\forall x \in I; \forall y \in f(I): y = f(x) \Leftrightarrow x = f^{-1}(y) \ldots . (*)$$

Remark

Relation $(*)$ is used to give the expression $f^{-1}(x)$ if it is possible.

Proof

If f is strictly monotonic over I, it is injective, and from the definition of the set $f(I)$, it is surjective, so f is bijective.

f is continuous, $f(I)$ is an interval. On the other hand, as f is strictly monotonic, f^{-1} is also monotonic. Therefore, f^{-1} is continuous because $f^{-1}(f(I)) = I$ is an interval (according to the theorem 4.14).

Example

Let the function f defined on the interval $\mathrm{I} = [0; +\infty[$ by $f(x) = x^2 + 3$, then f is continuous and strictly monotonic (increasing) on the interval $\mathrm{I} = [0; +\infty[$ where $f(\mathrm{I}) = [3; +\infty[$ according to the theorem (4.15), f is a bijective to the interval [0;+∞[in the interval [3;+∞[, so it accepts an inverse function f^{-1} and we have:

$\forall x \in [0; +\infty[; \forall y \in [3; +\infty[: y = x^2 + 3 \Leftrightarrow x^2 = y - 3$

$$\Leftrightarrow \begin{cases} x = \sqrt{y-3} \\ \qquad\qquad \vee \\ x = -\sqrt{y-3} < 0 (\text{unacceptable}). \end{cases}$$

So $f^{-1}(x) = \sqrt{y-3}$, after replacing x by y, the definition of the function f^{-1} becomes as follows:

$$f^{-1}: [3; +\infty[\rightarrow [0; +\infty[$$
$$x \rightarrow \sqrt{x-3}$$

Exercise*

Let the function f defined on $\mathbb{R}$ by $f(x) = \begin{cases} x^2 - 2x + 1 & si\ x \leq 1 \\ \frac{-x+1}{2x-1} & si\ x > 1 \end{cases}$.

1) Prove That f is continuous and strictly monotonic over $\mathbb{R}$.

2) Concluding that f accepts an inverse function f^{-1}, and write the expression $f^{-1}(x)$ in terms of x.

Solution

$\lim\limits_{\substack{> \\ x \to 1}} f(x) = \lim\limits_{\substack{< \\ x \to 1}} (x) = f(1) = 0 \Longrightarrow$ continuous at $0 \Longrightarrow f$ continuous over $\mathbb{R}$.

f is strictly decreasing over $\mathbb{R}$ and $f(\mathbb{R}) = \left]-\frac{1}{2}; +\infty\right[$. So

$$f^{-1}: \left]-\frac{1}{2}; +\infty\right[\to \mathbb{R}$$

$$x \to f(x) = \begin{cases} \dfrac{x+1}{2x+1} & \text{if } \dfrac{-1}{2} < x < 0 \\ 1-\sqrt{x} & \text{if } x \geq 0 \end{cases}$$

4.4 Differentiable functions

4.4.1 Definition and basic properties

Definition 4.15

Let f be a function defined on the neighborhood V_{x_0} of the point x_0. We say that the function f is differentiable at x_0 if and only if $\lim\limits_{\substack{> \\ x\to x_0}} \frac{f(x)-f(x_0)}{x-x_0} = \mathrm{L}$, exists. We call L the derivative of f at x_0, and we denote it by. $f'(x_0)$.

If f is differentiable in each point of I, then it is called differentiable on I, in this case we define the derivative function by $\begin{array}{c} f': I \to \mathbb{R} \\ x \to f'(x) \end{array}$. The derivative is sometimes written as $\frac{df}{dx}$ or $\frac{dy}{dx}$ where $y = f(x)$.

Remarks

1) By putting $x - x_0 = h$, the previous limit is written as $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$.

2) The function f is differentiable at x_0 if and only if there exists a function ε defined in the neighborhood V_{x_0} to the point x_0 where

$$\forall x \in V_{x_0}: f(x) - f(x_0) = \big(f'(x_0) + \varepsilon(x)\big)(x - x_0) \text{ و } \lim_{x\to x_0} \varepsilon(x) = 0$$

If $\lim\limits_{\substack{> \\ x\to x_0}} \frac{f(x)-f(x_0)}{x-x_0} = L_d$ ($\lim\limits_{\substack{< \\ x\to x_0}} \frac{f(x)-f(x_0)}{x-x_0} = L_g$, respectively), we say that the function f is differentiable at x_0 from the right (from the left, respectively) And we write $L_d = f'(x_0 + 0)$ ($L_g = f'(x_0 - 0)$, respectively).

Corollary 4.2

A function f is differentiable at x_0 if and only if $f'(x_0 - 0)$ and $f'(x_0 + 0)$ exist and

$f'(x_0 + 0) = f'\,(x_0 - 0)$.

Example

Let f be a function defined in $\mathbb{R}$ by $f(x) = |x^2 - 1|$, let us study the differentiability

of f at $x_0 = 1$.We have

$\lim\limits_{\substack{> \\ x\to 1}} \frac{f(x)-f(1)}{x-1} = \lim\limits_{\substack{> \\ x\to 1}} \frac{x^2-1}{x-1} = 2 = f'(1+0)$ and $\lim\limits_{\substack{< \\ x\to 1}} \frac{f(x)-f(1)}{x-1} = \lim\limits_{\substack{< \\ x\to 1}} \frac{-(x^2-1)}{x-1} = -2 = f'(1-0)$.

f is differentiable at $x_0 = 1$ from the right and from the left, but it is not differentiable at $x_0 = 1$ because $f'(1+0) \neq f'(1-0)$.

Geometric interpretation

The derivative of the function f at x_0 is the slope of the line tangent to the graph

of f at the point $\mathrm{M}_0(x_0, \mathrm{f}(x_0))$. Thus, the equation of this tangent line is

$y = f'(x_0)(x - x_0) + f(x_0)$.

The left and right derivatives are also interpreted by the half-tangents to the left and right of the point $\mathrm{M}_0(x_0, \mathrm{f}(x_0))$.

Theorem 4.16

If f is differentiable at x_0, then f is continuous at x_0.

Proof

Let f be differentiable at x_0 then there is a neighborhood V_{x_0} where

$\forall x \in V_{x_0}: f(x) - f(x_0) = \big(f'(x_0) + \varepsilon(x)\big)(x - x_0)$ and $\lim\limits_{x \to x_0} \varepsilon(x) = 0$.So

$\lim\limits_{x \to x_0} \big(f(x) - f(x_0)\big) = \lim\limits_{x \to x_0} \big(f'(x_0) + \varepsilon(x)\big)(x - x_0) = 0$ So f is continuous at x_0.

4.4.2 Higher order derivative

Let f be a function differentiable on the interval I. If f' differentiable on the interval I, then we denote its derivative by f'' and is called the second derivative. In the same way, we define the successive derivatives of the function f as follows:

$$\forall n \in \mathbb{N}:\ f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' \ \text{ and } \ f^{(0)}(x) = f(x),$$

Where $f^{(n)}$ symbolizes the nth order derivative of the function f, sometimes we denote $f^{(n)}$ by $\frac{d^n y}{dx^n}$ or $y^{(n)}$, where $y = f(x)$.

Exercise

Prove that:

1) $\forall n \in \mathbb{N}:\ cos^{(n)} x = \cos\left(x + \frac{\pi}{2} n\right)$.

2) $\forall n \in \mathbb{N}:\ \left[\frac{1}{x}\right]^{(n)} = \frac{(-1)^n n!}{x^{n+1}}$.

Definition 4.16

Let f be a function defined on the interval I.

We say that f is of a class C^n if it is differentiable to order n and the derivative $f^{(n)}$ is continuous over I. We denote the set of functions of class C^n on I by $.C^n(I)$.

By definition we have: $C^0(I) = C(I)$.

The set of infinitely differentiable functions on the interval I, symbolizes it by $C^\infty(I)$.

4.4.3 Operations on differentiable functions

Theorem 4.17

Let u and v be differentiable functions on the interval I, then the functions $u + v$, αu , $u.v$, $\frac{u}{v}$

($v \neq 0$) are differentiable over I and we have:

$$(u+v)' = u' + v' \quad , \quad (\alpha u)' = \alpha u'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad , \quad (uv)' = u'v + uv'.$$

Proof (Let us prove the last case)

Let $x_0 \in I$ we have

$$\frac{\frac{u}{v}(x) - \frac{u}{v}(x_0)}{x - x_0} = \frac{u(x)v(x_0) - u(x_0)v(x)}{v(x)v(x_0)(x - x_0)} = \frac{\frac{u(x)-u(x_0)}{(x-x_0)}v(x_0) - u(x_0)\frac{v(x)-v(x_0)}{(x-x_0)}}{v(x)v(x_0)}.$$

When $x \to x_0$ then $\frac{u(x)-u(x_0)}{(x-x_0)} \to u'(x_0)$; $\frac{v(x)-v(x_0)}{(x-x_0)} \to v'(x_0)$; $u(x) \to u(x_0)$ and

$v(x) \to v(x_0)$. So $\frac{\frac{u}{v}(x) - \frac{u}{v}(x_0)}{x - x_0} \longrightarrow \frac{u'(x_0)v(x_0) - u(x_0)v'(x_0)}{\left(v(x_0)\right)^2}$.

Theorem 4.18 (Leibniz formula)

If u and v admit nth order derivatives on the interval I then the function $u.v$ admits an nth order derivative on the interval I and we have:

$$\forall n \in \mathbb{N}: (u.v)^{(n)} = \sum_{p=0}^{n} C_n^p \, u^{(n-p)} v^{(p)}.$$

Proof

We use proof by induction and by noting that: $\forall n, p \in \mathbb{N}$ ($1 \leq p \leq n - 1$): $C_n^p = C_{n-1}^p + C_{n-1}^{p-1}$.

Theorem 4.19

Let u and v be functions where u is differentiable on the interval I and v is differentiable on the interval $u(I)$, then the function $v \circ u$ is differentiable on the interval I and $(v \circ u)' = v' \circ u \, . u'$.

Proof

Let $x_0 \in I$ since u is differentiable at x_0 and v is differentiable at $y_0 = u(x_0)$, Then

$$u(x) - u(x_0) = \left(u'(x_0) + \varepsilon_1(x)\right)(x - x_0) \ with \ \lim_{x \to x_0} \varepsilon_1(x) = 0$$

and

$$v(y) - v(y_0) = \left(v'(y_0) + \varepsilon_2(y)\right)(y - y_0) \text{ with } \lim_{y \to y_0} \varepsilon_2(y) = 0.$$

For $y = u(x)$ then $y \to y_0$ when $x \to x_0$ (since u is continuous at x_0) and from there

$v\big(u(x)\big) - v\big(u(x_0)\big) = \Big(v'\big(u(x_0)\big) + \varepsilon_2(y)\Big)\left(u'(x_0) + \varepsilon_1(x)\right)(x - x_0)$ and

$\dfrac{v\big(u(x)\big) - v\big(u(x_0)\big)}{x - x_0} = \Big(v'\big(u(x_0)\big) + \varepsilon_2(y)\Big)\left(u'(x_0) + \varepsilon_1(x)\right)$

For $x \to x_0$ then $y \to y_0$, $\varepsilon_1(x) \to 0$ and $\varepsilon_2(y) \to 0$.So

$$\frac{v\big(u(x)\big) - v\big(u(x_0)\big)}{x - x_0} \to v'\big(u(x_0)\big).u'(x_0).$$

Examples

1) Let the function f defined on $\mathbb{R}_+$ by $f(x) = \cos\left(3\sqrt{x} + x^2\right)$.

Putting $f = v \circ u$ where $\begin{cases} u(x) = 3\sqrt{x} + x^2 \\ v(x) = \cos x \end{cases}$, we have $\begin{cases} u'(x) = 3\sqrt{x} + x^2 \\ v'(x) = \cos x \end{cases}$.

So

$$f'(x) = (v' \circ u)(x).u'(x) = -\sin\left(3\sqrt{x} + x^2\right)\left(\frac{3}{2\sqrt{x}} + 2x\right)$$

$$= -\left(\frac{3}{2\sqrt{x}} + 2x\right)\sin\left(\sqrt{x} + x^2\right).$$

2) Let the function g defined by $g(x) = \ln\left(\sin\frac{x+1}{2x-3}\right)$.

Putting $g = v \circ u$ where $\begin{cases} u(x) = \sin\frac{x+1}{2x-3} \\ v(x) = \ln x \end{cases}$, we have $v'(x) = \frac{1}{x}$.

So

$$g'(x) = (v' \circ u)(x).u'(x) = \frac{1}{\sin\dfrac{x+1}{2x-3}}\ u'(x).$$

Next putting $u = v_1 \circ u_1$ where $\begin{cases} u_1(x) = \frac{x+1}{2x-3} \\ v_1(x) = \sin x \end{cases}$, we have $\begin{cases} u_1'(x) = -\frac{5}{(2x-3)^2} \\ v_1'(x) = \cos x \end{cases}$.

So

$$u'(x)(x) = (v_1' \circ u_1)(x).u_1'(x) = \cos\frac{x+1}{2x-3} \times \left(-\frac{5}{(2x-3)^2}\right).$$

So

$$g'(x) = \frac{1}{\sin\frac{x+1}{2x-3}} \cos\frac{x+1}{2x-3} \times \left(-\frac{5}{(2x-3)^2}\right) = -\frac{5}{(2x-3)^2} \operatorname{cotang}\frac{x+1}{2x-3}$$

Theorem 4.20

If f is a strictly monotonic continuous function on the interval I, and differentiable at x_0 in I where $f'(x_0) \neq 0$, then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$. And we have:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'[f^{-1}(y_0)]}.$$

Proof

Let f is differentiable at x_0 in I where $f'(x_0) \neq 0$, and let y_0 be a point in $f(I)$ where

$y_0 = f(x_0)$. For every y of $f(I)$ there is a single real number x of I where $y = f(x)$, since f is continuous and strictly monotonic on I, then f^{-1} is continuous and strictly monotonic on $f(I)$ (according to the Theorem 4.15), so $\forall y \in f(I): y \neq y_0 \Rightarrow x \neq x_0$.and for $y \longrightarrow y_0$, then $x \to x_0$.

Putting $g = f^{-1}$ then $y_0 = f(x_0) \Leftrightarrow x_0 = g(y_0)$ and $y = f(x) \Leftrightarrow x = g(y)$. So

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{x - x_0}{y - y_0} = \lim_{x \longrightarrow x_0} \frac{1}{\frac{y - y_0}{x - x_0}} = \frac{1}{f'(x_0)}.$$

Examples

1) Let $\begin{smallmatrix} f:[0;+\infty[\to\mathbb{R} \\ x\to f(x)=x^n \end{smallmatrix}$. The function f is continuous and strictly increasing on the interval $I = [0; +\infty[$, and from it, f accepts an inverse function f^{-1} defined, continuous and strictly increasing on the interval $f(I) = [0; +\infty[$, denoted by " $\sqrt[n]{.}$ " or " $(.)^{\frac{1}{n}}$ ", is called nth-root function.

Since: $\forall x \in]0, +\infty[: (x^n)' = nx^{n-1} \neq 0$, Then the function f^{-1} is differentiable at every number y where $y = x^n$ (i.e. f^{-1} is differentiable on the interval $]0, +\infty[$) and we have:

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n\left((y)^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}.$$

After changing x with y we get:

$$\forall x \in]0, +\infty[: \left(\sqrt[n]{x}\right)' = \left(x^{\frac{1}{n}}\right)' = \frac{1}{n} x^{\frac{1}{n}-1}.$$

2) Let $\underset{x \to h(x) = \tan x}{h: \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\to \mathbb{R}}$. The function h is continuous and strictly increasing on the interval $I = \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$, and from it, h accepts an inverse function h^{-1} defined, continuous and strictly increasing on the interval $h(I) = \mathbb{R}$, denoted by " arctan ". Since: $\forall x \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[: h'(x) = (\tan x)' = \frac{1}{cos^2 x} \neq 0$. Then the function h^{-1} is differentiable at every number y where $y = \tan x$ (i.e. h^{-1} is differentiable on $\mathbb{R}$) and we have: $(h^{-1})'(y) = \frac{1}{h'(x)} = cos^2 x = \frac{1}{1+tan^2 x} = \frac{1}{1+y^2}$.

After changing x with y we get:

$$\forall x \in \mathbb{R}: (\arctan x)' = \frac{1}{1+x^2}.$$

Theorem 4.21

If the function f has an extremum at point x_0 and is differentiable at x_0 then $f'(x_0) = 0$.

Proof

The existence of $f'(x_0)$ entails the existence and equality of $f'(x_0 + 0)$ and $f'(x_0 - 0)$. Assume that $f(x_0)$ is a maximum, then exists a neighbourhood V_{x_0} of the point x_0 where$\forall x \in V_{x_0}$: $f(x) \leq f(x_0)$. So if $x > x_0$ then $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$ and if $x < x_0$ then $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. So

$f'(x_0) = f'(x_0 + 0) = \lim\limits_{\substack{x \to x_0 \\ <}} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$ and

$f'(x_0) = f'(x_0 - 0) = \lim\limits_{\substack{x \to x_0 \\ >}} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$.

This implies that $f'(x_0) = 0$

4.4.4 Mean value theorem

4.4.4.1 Rolle's Theorem

Theorem 4 22

If a function f $[a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$ and $f(a) = f(b)$, then there exists a point c in $]a, b[$ such that $f'(\text{c}) = 0$.

Proof

Since the function f is continuous on $[a, b]$, there exist a points x_{m} and x_{M} in $[a, b]$ where f take their minimum and maximum values respectively.

If $f(x_{\text{m}}) = f(x_{\text{M}})$ then the function f is constant on $[a, b]$ so in this case we have: $\forall x \in]a; b[: f'(x) = 0$.

If $f(x_{\mathrm{m}}) < f(x_{\mathrm{M}})$ then since $f(a) = f(b)$ one of the two points x_{m} and x_{M} belongs to the open interval $]a, b[$. We denote it by c. According to the theorem 4.21 we obtain $f'(\mathrm{c}) = 0$.

4.4.4.1 Mean value theorem

Theorem 4 23 (Lagrange's theorem)

If a function $f\ [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, then there exists a point $c \in]a, b[$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof

It suffices to verify that the function g, defined on the interval $[a, b]$ by $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$, satisfies the conditions of Theorem 4.22. Then there is at least one number c in the interval $]a, b[$which satisfies $g'(c) = 0$ therefore $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Remark

This theorem is used in approximate calculations and in proving many inequalities.

Example

Using the mean value theorem, prove that: $\forall x \geq 0$: $\frac{x}{1+x} \leq \ln(x+1) \leq x$.

Answer

By applying the mean value theorem to the function $f(x) = \ln(x+1)$ on the interval $\left[\underbrace{0}_{a}; \underbrace{x}_{b}\right]$ where $x \geq 0$, we get:

$$\forall x \geq 0 : \underbrace{\ln(x+1)}_{f(b)} - \underbrace{\ln 1}_{f(a)} = f'(c)\left(\underbrace{x}_{b} - \underbrace{0}_{a}\right) \quad , \quad \underbrace{0}_{a} < \mathrm{c} < \underbrace{x}_{b}.$$

So

$$\ln(x+1) = f'(c)x = \frac{1}{1+c} \cdot x \quad , \quad 0 < \mathrm{c} < x.$$

We have

$$0 < \mathrm{c} < x \implies \frac{1}{1+x} < \frac{1}{1+c} < 1 \implies \frac{x}{1+x} \leq \frac{1}{1+c}x \leq x.$$

We obtain

$$\forall x \geq 0 : \frac{x}{1+x} \leq \ln(x+1) \leq x.$$

For example if $x = 0.02$ then $0.0196 \leq \frac{0.02}{1.02} \leq \ln(1.02) \leq 0.02$.

4.4.4.3 Generalized mean value theorem

Theorem 4 24 (Cauchy's theorem)

If a functions $f, g\ [a,b] \rightarrow \mathbb{R}$ are continuous on the closed interval $[a,b]$ and differentiable on the open interval $]a,b[$, and g' is non-zero in the interval $]a,b[$ then there exists a point $c \in]a,b[$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof

We have $(\forall x \in]a\,;b[: g'(x) \neq 0) \Longrightarrow \big(g(b) \neq g(a)\big)$, so it is suffices to verify that the function φ, defined on the interval $[a,b]$ by $\varphi(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g(x)$, satisfies the conditions of theorem 4.22. Then there is at least one number c in the interval $]a,b[$ which satisfies $\varphi'(c) = 0$ therefore $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{b-a}$.

Theorem 4 25 (Hospital Rule)

Let f and g be a continuous functions on a neighbourhood V_a of the point a and differentiable on $V - \{a\}$ then:

If the $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then the $\lim\limits_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$ exists also and $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$.

In particular if $f(a) = g(a) = 0$ we have $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f(x)}{g(x)}$.

Proof

Assume that $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \ell$.

If $x > a$ (If $x < a$, respectively) by applying the theorem 4 24 on the interval $[a,x]$ (on the interval $[x,a]$ respectively) we get:

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)} \text{ where } c \text{ between } a \text{ and } x.$$

So $(x \to a) \Longrightarrow (c \to a) \Longrightarrow \frac{f'(c)}{g'(c)} \to \ell \Longrightarrow \frac{f(x)-f(a)}{g(x)-g(a)} \to \ell$. So $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f(x)}{g(x)} = \ell$.

Remarks

1) The Hospital Rule remains true if f and g are not defined in a, but have two finite limits.

2) The Hospital Rule can be applied several times in a row.

3) The Hospital Rule can be applied in the following cases:

a) $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 0$.

b)) $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$.

c)) $\lim_{x\to \infty} f(x)=\infty$ and $\lim_{x\to \infty} g(x)=\infty$.

Examples

1) $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$ (I.F $\frac{0}{0}$).

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\lim_{x\to 1}\frac{\frac{1}{2\sqrt{x+3}}}{1}=\frac{1}{4}.$$

2) $\lim_{x\to 0}\frac{e^x-x-1}{x^2}$ (I.F $\frac{0}{0}$).

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}.$$

3) $\lim_{x\to +\infty}\frac{e^x+x^2}{x^3-x+1}$ (I.F $\frac{\infty}{\infty}$).

$$\lim_{x\to +\infty}\frac{e^x+x^2}{x^3-x+1}=\lim_{x\to +\infty}\frac{e^x+2x}{3x^2-1}=\lim_{x\to +\infty} e^x\frac{e^x+1}{6x}=\lim_{x\to +\infty}\frac{e^x}{6}=+\infty.$$

4) $\lim_{x\to +\infty}\frac{2x^2}{x+3}\ln\frac{x-1}{x+2}$ (I.F $\infty.0$)

$$\lim_{x\to +\infty}\frac{2x^2}{x+3}\ln\frac{x-1}{x+2}=\lim_{x\to +\infty}\frac{2x}{x+3}\lim_{x\to +\infty}\frac{\ln\frac{x-1}{x+2}}{\frac{1}{x}}.$$

Calculate $\lim_{x\to +\infty}\frac{\ln\frac{x-1}{x+2}}{\frac{1}{x}}$ (I.F $\frac{0}{0}$).

$$\lim_{x\to +\infty}\frac{\ln\frac{x-1}{x+2}}{\frac{1}{x}}=\lim_{x\to +\infty}\frac{\left(\ln\frac{x-1}{x+2}\right)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to +\infty}\frac{\frac{3}{(x+2)(x-1)}}{-\frac{1}{x^2}}=-3$$

So $\lim_{x\to +\infty}\frac{2x^2}{x+3}\ln\frac{x-1}{x+2}=2\times(-3)=-6.$

Chapter Five: Elementary functions

5.1 Inverse Trigonometric functions

5.1.1 Arcsine Function

Definition 5.1

The function f defined in the interval $I = \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ by $f(x) = \sin x$, is continuous and strictly increasing in the interval I, it accepts an inverse function f^{-1} that is defined, continuous and strictly increasing on the interval $f(I) = [-1;1]$. We denote the function f^{-1} by "arcsin" or "$\sin^{-1}$". And we have $\forall x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]; \forall y \in [-1;1] : y = \sin x \Leftrightarrow x = \arcsin y$.

Derived Function

We have $\forall x \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[: f'(x) = (\sin x)' = \cos x \neq 0$.

According to the Theorem 4.20 then, the function " arcsin " is differentiable at every number y where $y = \sin x$ (i.e. on the interval $]-1;1[$) and we have:

$$\begin{aligned}[f^{-1}(y)]' &= \frac{1}{f'(x)} \\ &= \frac{1}{\cos x} \\ &= \frac{1}{\sqrt{1-\sin^2 x}} \begin{pmatrix}\text{Since } \cos^2 x + \sin^2 x = 1, \text{ and} \\ x \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\Rightarrow \cos x > 0\end{pmatrix} \\ &= \frac{1}{\sqrt{1-y^2}}.\end{aligned}$$

After changing x with y we get:

$$\forall x \in]-1;1[: (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

5.1.2 Arccosine Function

Definition 5.2

The function g defined in the interval $I = [0;\pi]$ by $g(x) = \cos x$, is continuous and strictly decreasing in the interval I, it accepts an inverse function g^{-1} that is defined, continuous and strictly decreasing on the interval $g(I) = [-1;1]$. We denote the function g^{-1} by "arccos" or "$\cos^{-1}$". And we have $\forall x \in [0;\pi]; \forall y \in [-1;1] : y = \cos x \Leftrightarrow x = \arccos y$.

Derived Function

We have $\forall x \in]0;\pi[: g'(x) = (\cos x)' = -\sin x \neq 0$.

Then the function " arccos " is differentiable at every number y where $y = \cos x$ (i.e. on the interval $]-1;1[$) and we have:

$$\begin{aligned}[g^{-1}(y)]' &= \frac{1}{g'(x)} \\ &= \frac{1}{-\sin x} \\ &= \frac{1}{-\sqrt{1-\sin^2 x}} \quad \begin{pmatrix}\text{Since } \cos^2 x + \sin^2 x = 1, \text{ and} \\ x \in]0;\pi[\Rightarrow \sin x > 0\end{pmatrix} \\ &= \frac{1}{-\sqrt{1-y^2}}.\end{aligned}$$

After changing x with y we get:

$$\forall x \in]-1;1[: (\arccos x)' = \frac{-1}{\sqrt{1-x^2}}.$$

5.1.3 Arctangent Function

Definition 5.3

The function h defined in the interval $I = \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$by $h(x) = \tan x$, is continuous and strictly increasing in the interval I, it accepts an inverse function h^{-1} that is defined, continuous and strictly increasing on the interval $h(I) = \mathbb{R}$. We denote the function h^{-1} by " arctan " or "$\tan^{-1}$".

And we have $\forall x \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[; \forall y \in \mathbb{R} : y = \tan x \Leftrightarrow x = \arctan y$.

Derived function

We have $\forall x \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[: h'(x) = (\tan x)' = \frac{1}{\cos^2 x} \neq 0$

Then, the function " arctan " is differentiable at every number y where $y = \tan x$ (i.e. on $\mathbb{R}$) and we have:

$$[g^{-1}(y)]' = \frac{1}{g'(x)}$$
$$= \cos^2 x$$
$$= \frac{1}{1 + \tan^2 x}$$
$$= \frac{1}{1 + y^2}.$$

After changing x with y we get:

$$\forall x \in \mathbb{R} : (\arctan x)' = \frac{1}{1 + x^2}.$$

5.1.4 Arccotangent Function

Definition 5.4

The function k defined in the interval $I =]0; \pi[$ by $k(x) = \text{cotan}\, x$, is continuous and strictly decreasing in the interval I, it accepts an inverse function k^{-1} that is defined, continuous and strictly decreasing on the interval $k(I) = \mathbb{R}$. We denote the function k^{-1} by " arccotan " or " cotan^{-1} ". And we have $\forall x \in]0; \pi[; \forall y \in \mathbb{R} : y = \text{cotan}\, x \Leftrightarrow x = \text{arccotan}\, y$.

Derived function

Similarly we have

$$\forall x \in \mathbb{R} : (\text{arccotan}\, x)' = -\frac{1}{1 + x^2}.$$

Properties

1) $\forall x \in [-1; 1] : \arcsin x + \arccos x = \frac{\pi}{2}$.

2) $\forall x \in [-1 ; 1] : \sin(\arccos x) = \sqrt{1 - x^2}$.

3) $\forall x \in [-1; 1] : \cos(\arcsin x) = \sqrt{1 - x^2}$.

4) $\forall x \in \mathbb{R} : \text{arc}\tan x + \text{arc cotan}\, x = \frac{\pi}{2}$.

5) $\forall x > 0 : \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$.

6) $\forall x < 0 : \arctan x + \arctan \frac{1}{x} = -\frac{\pi}{2}$.

Proof

1) We put $\forall x \in [-1; 1]: f(x) = \arcsin x + \arccos x$.

We have $\forall x \in]-1;1[: f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$. So the function f is constant in the interval $[-1;1]$. So $\forall x \in [-1;1]: f(x) = f(0) = \frac{\pi}{2}$.

2) We have $\forall x \in [-1;1]$: $\arcsin x \; x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \Longrightarrow \cos(\arcsin x) \geq 0$. So

$\forall x \in [-1;1]: \cos(\arcsin x) = \sqrt{1 - \left(\sin\,(\arcsin x)\right)^2} = \sqrt{1-x^2}$.

6) We put $\forall x < 0 : f(x) = \arctan x + \arctan \frac{1}{x}$. We have

$\forall x < 0 : f'(x) = \frac{1}{1+x^2} - \frac{1}{x^2}\frac{1}{1+\left(\frac{1}{x}\right)^2} = 0$. So the function f is constant in the interval $]-\infty;0[$. So $\forall x \in]-\infty;0[: f(x) = f(-1) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$.

Remark

The properties of inverse trigonometric functions are deduced from the properties of trigonometric functions. For example, property 1 is deduced from the property: $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$, which we will explain later.

We have $\frac{\pi}{2} - \alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \Leftrightarrow \alpha \in [0,\pi]$.

By putting $\cos\alpha = x$ we get $\alpha = \arccos x$ and we have

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \Leftrightarrow \sin\left(\frac{\pi}{2} - \alpha\right) = x$$

$$\Leftrightarrow \frac{\pi}{2} - \alpha = \arcsin x$$

$$\Leftrightarrow \frac{\pi}{2} - \arccos x = \arcsin x$$

$$\Leftrightarrow \frac{\pi}{2} = \arccos x + \arcsin x.$$

5.2 Hyperbolic functions and their inverses

5.2.1 Hyperbolic functions

Definition 5.5 The hyperbolic sine function, which we denote by “sh,” is defined as

$$\forall x \in \mathbb{R}: \operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

Definition 5.6 The hyperbolic cosine function, which we denote by “ch,” is defined as

$$\forall x \in \mathbb{R}: \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

Definition 5.7 The hyperbolic tangent function, which we denote by “th,” is defined as

$$\forall x \in \mathbb{R}: \text{th } x = \frac{sh\ x}{ch\ x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Definition 5.8 The hyperbolic cotangent function, which we denote by “th,” is defined as

$$\forall x \in \mathbb{R}^*: \coth x = \frac{ch\ x}{sh\ x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Properties

For all $x, y \ \in \mathbb{R}$ we have:

1) $sh\ (-x) = -sh\ x$ ، $ch\ (-x) = ch\ x$.

2) $1 - th^2\ x = \frac{1}{ch^2 x}$ ، $\text{ch}^2 x - sh^2 x = 1$.

3) $ch(x + y) = ch\ x\ ch\ y + sh\ x\ shy$.

4) $sh(x + y) = ch\ x\ sh\ y + sh\ x\ ch\ y$.

5) $th(x + y) = \frac{th\ x + th\ y}{1 + th\ x\ th\ y}$.

6) $(sh\ x)' = ch\ x$, $(ch\ x)' = sh\ x$, $(th\ x)' = \frac{1}{ch^2 x}$, $(coth\ x)' = -\frac{1}{sh^2 x}$.

5.2.2 Inverses Hyperbolic functions

Definition 5.9

The function f defined in the interval $I = [0; +\infty[$ by $f(x) = \text{ch}\, x$, is continuous and strictly increasing in the interval I, it accepts an inverse function f^{-1} that is defined, continuous and strictly increasing on the interval $f(I) = [1; +\infty[$. We denote the function f^{-1} by " arg ch " or "ch^{-1}".And we have:

$$\forall x > 0; \forall y > 1 : y = \text{ch}\, x \Leftrightarrow ch\ x = \frac{e^x + e^{-x}}{2}$$

$$\Leftrightarrow e^{2x} - 2ye^x + 1 = 0.$$

$$\Leftrightarrow \begin{cases} x = \ln\left(y + \sqrt{y^2 - 1}\right) \\ x = \ln\left(y - \sqrt{y^2 - 1}\right) \end{cases}$$

$$\Leftrightarrow x = \ln\left(y - \sqrt{y^2 - 1}\right)\left(\text{because } \ln\left(y - \sqrt{y^2 - 1}\right) \leq 0\right).$$

After changing x with y we get:

$$\forall x \geq 1 : \arg \operatorname{ch} x = \ln(x + \sqrt{x^2 - 1}).$$

Derived Function: $\forall x \in]1; +\infty[: (\arg \operatorname{ch} x)' = \frac{1}{\sqrt{x^2-1}}$.

Definition 5.10

The function g defined in the interval $I = \mathbb{R}$ by $g(x) = \operatorname{sh} x$, is continuous and strictly increasing in the interval I, it accepts an inverse function g^{-1} that is defined, continuous and strictly increasing on the interval $f(I) = \mathbb{R}$. We denote the function g^{-1} by " $\arg \operatorname{sh}$ " or " sh^{-1} ". And we have:

$$\forall x \in \mathbb{R} : \arg \operatorname{sh} x = \ln(x + \sqrt{x^2 + 1}).$$

Derived function

$$\forall x \in \mathbb{R} : (\arg \operatorname{sh} x)' = \frac{1}{\sqrt{x^2 + 1}}.$$

Definition 5.11

The function h defined in the interval $I = \mathbb{R}$ by $h(x) = \operatorname{th} x$, is continuous and strictly increasing in the interval I, it accepts an inverse function h^{-1} that is defined, continuous and strictly increasing on the interval $h(I) =]-1; 1[$. We denote the function h^{-1} by " $\arg \operatorname{th}$ " or " th^{-1} ". And we have:

$$\forall x \in]-1; 1[: \arg \operatorname{th} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}.$$

Derived function

$$\forall x \in]-1; 1[: (\arg \operatorname{th} x)' = \frac{1}{1 - x^2}.$$