Semestre: 01

Unité d’enseignement : Fondamentale

Matière: Analyse1

Crédits: 6

Coefficient: 4

Objectifs de l’enseignement:

Approfondissement de la notion de fonctions de R dans R.

Connaissances préalables recommandées

Principes des mathématiques (Notions d’analyse classique)

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<u>Analysis 1 program</u>

Chapter one: The set of real numbers

Chapter Tow: Complex numbers

Chapter Three: Real sequences

Chapter Four: Real functions with real variable

Chapter five: Elementary functions

...

Chapter one: The set of real numbers

1 1.Algebraic structure of the set $\mathbb{R}$

The set of real numbers is a set that we denote by $\mathbb{R}$ equipped with the operation of addition (+) and multiplication (·) and an total ordering relation" $\leq$ "satisfies the following Axioms.

A1) $\forall x, y, z \in \mathbb{R}: x + (y + z) = (x + y) + z$.

A2) $\forall x, y \in \mathbb{R}: x + y = y + x$.

A3) $\forall x \in \mathbb{R}: x + 0 = 0 + x = x$.

A4) $\forall x \in \mathbb{R}: x + (-x) = (-x) + x = 0$.

A5) $\forall x, y, z \in \mathbb{R}: x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

A6) $\forall x, y \in \mathbb{R}: x \cdot y = y \cdot x$.

A7) $\forall x \in \mathbb{R}: x \cdot 1 = 1 \cdot x = x.$

A8) $\forall x \in \mathbb{R}^*: x \cdot x^{-1} = x^{-1} \cdot x = 1.$

A9) $\forall x, y, z \in \mathbb{R}: x \cdot (y + z) = x \cdot y + x \cdot z.$

A10) $\forall x \in \mathbb{R}: x \leq x.$

A11) $\forall x, y, z \in \mathbb{R}: \big(x \leq y$ و $y \leq z\big) \Rightarrow (x \leq z).$

A12) $\forall x, y \in \mathbb{R}: (x \leq y \text{ and } y \leq x) \Rightarrow (x = y).$

A13) $\forall x, y \in \mathbb{R}: x \leq y \text{ or } y \leq x.$

A14) $\forall x, y, z \in \mathbb{R}: (x \leq y) \Leftrightarrow (x + z \leq y + z).$

A15) $\begin{cases} \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_+^*: (x \leq y) \Leftrightarrow (x \cdot z \leq y \cdot z) \\ \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_-^*: (x \leq y) \Leftrightarrow (x \cdot z \geq y \cdot z) \end{cases}.$

Properties

1) $\forall x, y, x', y' \in \mathbb{R}: \big(x \leq y$ و$x' \leq y'\big) \Rightarrow (x + x' \leq y + y').$

2) $\forall x, y, x', y' \in \mathbb{R}_+^*: \big(x \leq y$ و$x' \leq y'\big) \Rightarrow (x \cdot x' \leq y \cdot y').$

4) $\forall x, y \in \mathbb{R}_+^*: (0 < x < y) \Leftrightarrow \left(0 < \frac{1}{y} < \frac{1}{x}\right).$

1.2 Absolute value

Definition 1.1 let it be $x \in \mathbb{R}$

The absolute value of the real number x is the positive real number which we denote by $|x|$and defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Properties : $x. y\ r.$ is a real numbers where $r \geq 0$

1) $|x| \geq 0;\ |-x| = |x|\ ; -|x| \leq x \leq |x|$

2)$|x| = 0 \Leftrightarrow x = 0$

3) $|x.y| = |x||y|$

4) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ (y \neq 0)$

5) $|x + y| \leq |x| + |y|$

6) $|x| \leq r \Leftrightarrow -r \leq x \leq r$

7) $|x| \geq r \Leftrightarrow x \leq -r\ or x \geq r$

***1.3.*Bounded subset in $\mathbb{R}$**

Definition 1.2

Let A be a non-empty sub set of $\mathbb{R}$.

- *We say that A is bounded from above if and only if:*

$$\exists b \in \mathbb{R}\ ;\ \forall x \in A : x \leq b$$

The number b is called upper bound of A

- *We say that A is bounded from below if and only if*

$$\exists a \in \mathbb{R}\ ;\ \forall x \in A : x \geq a$$

The number a *is called lower bound of* A.

A is bounded if and only if it is bounded from above and below.

Proposition 1.1 *The three following conditions are equivalent*

1*).A is bounded*

2) $\exists a \in \mathbb{R}\ ;\ \exists b \in \mathbb{R} : \forall x \in A : a \leq x \leq b$.

3) $\exists M \in \mathbb{R}_+^*\ ;\ \forall x \in A : |x| \leq M$

1.3.1 Suppremum, infimum,.maximum and minimum

The least upper bound from A is called supremum of A and denote it by sup A.

The greatest lower bound from A is called infimum of A and denote it by inf A.

If $supA \in A$ *is called maximum of* A *and denote it by* $\max A$.

If $infA \in A$ *is called minimum of* A *and denote it by* $\min A$.

Note

If A is not bounded above (below, respectively) in $\mathbb{R}$ we write $supA = +\infty$

($infA = -\infty$, *respectively).*

proposition 1.2

1)Let A *be bounded from above, then*

$$M = \sup A \Leftrightarrow \begin{cases} \forall x \in A : x \leq M \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists a \in A : M - \varepsilon < a \end{cases}$$

2)Let A *be bounded from below, then*

$$m = inf\ A \Leftrightarrow \begin{cases} \forall x \in A : x \geq m \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in A : m + \varepsilon > b \end{cases}$$

Proof

1) M is the smallest of the upper bounds if and only if the following proposition is false .

$$\exists M' < M ; \forall x \in A : x \leq M'$$

So if the proposition $\forall M' < M ; \exists x \in A : x > M'$, *is true.*

By putting $\varepsilon = M - M'$ *(*$\varepsilon > 0$*) so, the last proposition is written in the form:*

$$\forall \varepsilon > 0 \; ; \; \exists x \in A : M - \varepsilon < x.$$

2) In the same way we prove the second case

Example

Let $A = [1,2[$ *;* $maxA =$ *unvailable ;*$supA$ *=2 ;* i $infA = 1$ $minA = 1$

1.3.2* The Completeness axiom*:

Every nonempty subset of real numbers that is bounded from above has a **supremum**, and every nonempty subset of real numbers that is bounded from below has an **infimum**.

1.4 Archimedean axiom

Theorem 1.1:$\forall x > 0; \; \forall y \in \mathbb{R} \,; \exists n \in \mathbb{N}^* : y < nx.$

Proof: By contradiction

Suppose that: $\exists x > 0; \; \exists y \in \mathbb{R} \,; \forall n \in \mathbb{N}^* : y \geq nx$ or $\exists x > 0; \; \exists y \in \mathbb{R} \,; \forall n \in \mathbb{N}^* : n \leq \frac{y}{x}$,

then $\frac{y}{x}$ is an upper bound for $\mathbb{N}^*$; hence by the completeness axiom, $M = \sup \mathbb{N}^*$ exists.

So

$\forall \varepsilon > 0 \; ; \; \exists n_0 \in \mathbb{N}^* : M - \varepsilon < n_0$ and by putting $\varepsilon = 1$,we get $\exists n_0 \in \mathbb{N}^* : M < n_0 + 1 \in \mathbb{N}^*$; contradicting the fact that M is an upper bound for $\mathbb{N}^*$.

1.5 The integer part of a real number

For every real number x there is a unique integer number which we denote as $\mathrm{E}(x)$ or $[x]$, such that $\mathrm{E}(x) \leq x < \mathrm{E}(x) + 1$.

$\mathrm{E}(x)$ is called the integer part of the real number x.

In other words $\mathrm{E}(x)$ is the largest integer less than or equal to x.

<u>Examples</u>

1) For $x = 0.13$*, suppose* $E(x) = n$*.*
So
$(n \leq 0.13 < n + 1$ *where* $n \in \mathbb{Z}) \Longrightarrow \; n = 0.$
So

$$E(0.13) = 0.$$

2) For $x = -0.13$*, suppose* $E(x) = m$*.*
So
$(m \leq -0.13 < m + 1$ *where* $m \in \mathbb{Z}) \Longrightarrow \; m = -1.$
So

$$E(-0.13) = -1.$$

Solved exercises

1) Let A be a subset of real numbers where $A = \left\{\frac{1}{n}; \; n \in \mathbb{N}^*\right\}$.

Specify if possible $supA$, $maxA$, $infA$, $minA$.

Solution

We have $\forall n \in \mathbb{N}^*: n \geq 1 \Longrightarrow 0 < \frac{1}{n} \leq 1$ so the subset A is bounded, according to completeness axiom $supA$ and $infA$ exists.

Now we have $1 \in A$ so $maxA = supA = 1$.

The number 0 is an upper bound for A and $0 \notin A$, let we prove that $infA = 0$.

For this we will show that $\forall \varepsilon > 0;\ \exists b \in A: 0 + \varepsilon > b$ or $\forall \varepsilon > 0;\ \exists n \in \mathbb{N}^*: \varepsilon > \frac{1}{n}$ or $\forall \varepsilon > 0;\ \exists n \in \mathbb{N}^*: 1 < \varepsilon n$, this last proposition is true according to Archimedean axiom.

2) a) Let A and B be non-empty bounded subsets of real numbers. The set $A - B$ is defined as $A - B = \{x - y : x \in A, y \in B\}$. Prove that $Sup(A - B) = SupA - InfB$. and

$Inf(A - B) = InfA - SupB$.

b) Find the infimum and supremum of the subset $T = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}^*\right\}$.

Solution

a) We have

$$M = \sup A \Leftrightarrow \begin{cases} \forall x \in A : x \leq M \dots\dots\dots\dots\dots\dots\dots\dots (1) \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists a \in A : M - \dfrac{\varepsilon}{2} < a \dots\dots (2) \end{cases}$$

also

$$m = inf\ B \Leftrightarrow \begin{cases} \forall y \in B : y \geq m \dots\dots\dots\dots\dots\dots\dots\dots (3) \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in B : m + \dfrac{\varepsilon}{2} > b \dots\dots\dots (4) \end{cases}.$$

$$\Leftrightarrow \begin{cases} \forall y \in B : -y \leq -m \dots\dots\dots\dots\dots\dots\dots\dots (5) \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in B : -m - \dfrac{\varepsilon}{2} < -b \dots\dots\dots (6) \end{cases}$$

By adding the inequalities (1) and (5) as well as the inequalities (2) and (6) we get

$$\begin{cases} \forall x \in A\ ; \forall y \in B: x - y \leq M - m \dots\dots\dots\ \dots\dots\dots\dots\dots (7) \\ \text{and} \\ \forall \varepsilon > 0\ ; \exists a \in A;\ \exists b \in B : M - m - \dfrac{\varepsilon}{2} < a - b \dots\dots\dots (6) \end{cases}.$$

Thus $Sup(A - B) = M - m = SupA - InfB$.

Similarly, we can prove that: $Inf(A - B) = InfA - SupB$.

b) We put $S = \left\{\frac{1}{n} : n \in \mathbb{N}^*\right\}$ and by the exercise 1 we have $supS = 1$ and $infS = 0$.

So also we have $T = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}^*\right\} = S - S$, and by the question a) we have

$$Sup(T\ ) = SupS - InfS = 1 - 0 = 1.$$

$$Inf(T\ ) = InfS - SupS = 0 - 1 = -1.$$

1.6 dense groups in $\mathbb{R}$

Theorem 1.2 ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Between any two distinct real numbers there is an rational number.

Proof

Let x and y be two real numbers where $x < y$ so $y - x > 0$.

According to Archimedean axiom, $\exists n \in \mathbb{N}^*: 1 < n(y - \ x)$ or $nx + 1 < ny$.

On the other hand we have $E(nx) \leq nx < E(nx) + 1$.

So

$$nx < E(nx) + 1 \leq nx + 1 < ny.$$

So

$$nx < E(nx) + 1 < ny$$

then

$$x < \frac{E(nx) + 1}{n} < y.$$

It then follows that the rational number $r = \frac{E(nx)+1}{n}$ *satisfies* $x < r < y$.

Definition 1.3 I

Irrational numbers are real numbers that are not rational numbers and are symbolized by I *or* $\mathbb{R}/\mathbb{Q}$.

proposition *1.3*

The number $\sqrt{2}$ *is an irrational number.*

Proof

Assume that $\sqrt{2} \in \mathbb{Q}$. Then let $\frac{p}{q} = \sqrt{2}$ where $p, q \in \mathbb{N}^*$ and $gcd(p.q) = 1$.

Then $\frac{p}{q} = \sqrt{2} \Longrightarrow p = q\sqrt{2} \Longrightarrow p^2 = 2q^2 \Longrightarrow q^2$ divide p^2.

Since $q^2$and p^2prime $\Longrightarrow q^2$ divide $1 \ \Longrightarrow \ q = 1$. By substitution in the previous equality we get $p^2 = 2$ and this is a contradiction because there is no natural number squared equal to 2.

proposition *1.4*

if $x \in \mathrm{I}$ and $\mathrm{r} \in \mathrm{Q}*$ then $\mathrm{r}x \in \mathrm{I}$.

Proof

Assume that $x \in I$ *and* $r \in \mathbb{Q}^*$ *and that* $rx \in \mathbb{Q}$ *and from him:*

$$\left(\frac{1}{r} \in \mathbb{Q}^* or\ rx \in \mathbb{Q}\right) \Rightarrow \frac{1}{r} rx \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$$

This is a contradiction because $x \in I$.

Theorem 1.3

Between any two distinct real numbers there is an irrational number.

Proof

Let x, y *be a real numbers, where* $x < y$*, according to the theorem 1.2, there is a rational number* r ($r \neq 0$) such that: $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$ or $x < r\sqrt{2} < y$ and according to propositions 1.3 and 1.4 we conclude that $r\sqrt{2}$ is a irrational number.

Corollary 1.1 The two sets $\mathbb{Q}$ and I is dense in $\mathbb{R}$.

1.7 Intervals in $\mathbb{R}$

Let a, b *a real numbers, where* $a < b$*, we define*

$[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$ *is called closed interval.*

$]a, b[= \{x \in \mathbb{R}: a < x < b\}$ *is called open interval.*

$[a, b[= \{x \in \mathbb{R}: a \leq x < b\}$ *is called half open interval.*

$]a, b] = \{x \in \mathbb{R}: a < x \leq b\}$ "

$[a, +\infty[= \{x \in \mathbb{R}: x \geq a\}$ *unbounded closed interval.*

$]-\infty, b] = \{x \in \mathbb{R}: x \leq b\}$ "

$]a, +\infty[= \{x \in \mathbb{R}: x > a\}$ *unbounded open interval.*

$]-\infty, b[= \{x \in \mathbb{R}: x < b\}$ "

$\mathbb{R} =]-\infty, +\infty[$ "

Theorem 1.4

The nonempty subset I of $\mathbb{R}$ *is an interval if and only if the following property is satisfied:*

$$\forall a, b \in I\ (a \leq b);\ \forall x \in \mathbb{R}: a \leq x \leq b \Rightarrow x \in I$$

Proof

(⇐)Necessary condition: It is a clear that: if the set I is a interval, then the property is true.

(⇒)Sufficient condition: If the property is true, then the set I is a interval.

We have four possible cases, case 1: I is bounded, case 2: I is bounded from above and unbounded from below, case 3: I is bounded from below and unbounded from above, case 4: I is neither bounded from above nor from below.

Let us prove that in the first case then: $I = [a, b]$ or $I = [a, b[$ or $I =]a, b]$ or $I =]a, b[$ where $a = inf\ I$ and $b = sup\ I$.

We have: $$b = sup\ I \Leftrightarrow \begin{cases} \forall x \in I : x \leq b \\ \text{و} \\ \forall \varepsilon > 0 ;\ \exists b' \in I : b - \varepsilon < b' \dots\dots (1) \end{cases}.$$

and

$$a = inf\ I \Leftrightarrow \begin{cases} \forall x \in I : x \geq a \\ \text{و} \\ \forall \delta > 0 ;\ \exists a' \in I : a + \delta > a' \dots\dots (2) \end{cases}.$$

case 1: *If $a \in I$ and $b \in I$, then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x \leq b \Rightarrow x \in [a, b] \Rightarrow I \subset [a, b]$$

$$\forall x \in \mathbb{R}: x \in [a, b] \Rightarrow a \leq x \leq b \Rightarrow x \in I \Rightarrow [a, b] \subset I$$

So

$$I = [a, b].$$

case 2: *If $a \in I$ and $b \notin I$, then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x < b \Rightarrow x \in [a, b[\Rightarrow I \subset [a, b[$$

$$\forall x \in \mathbb{R}: x \in [a, b[\Rightarrow a \leq x < b \Rightarrow b - x > 0$$

putting $\varepsilon = b - x$ in (1) *we get $x < b'$ and since $a, b' \in I$, then:*

$$a \leq x < b' \Rightarrow x \in I \Rightarrow [a, b[\subset I$$

so

$$I = [a, b[.$$

case 3: *If $a \notin I$ and $b \in I$, then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x \leq b \Rightarrow x \in]a, b] \Rightarrow I \subset]a, b]$$

$$\forall x \in \mathbb{R}: x \in]a, b] \Rightarrow a < x \leq b \Rightarrow x - a > 0$$

By putting $\delta = x - a$ in (2)we get $x > a'$and since $a, a' \in I$, then:

$$a' < x \leq b \Rightarrow x \in I \Rightarrow]a, b] \subset I.$$

So

$$I =]a, b]$$

case 4: *If $a \notin I$ and $b \notin I$, Then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x < b \Rightarrow x \in]a, b[\Rightarrow I \subset]a, b[$$

$$\forall x \in \mathbb{R}: x \in]a, b[\Rightarrow a < x < b \Rightarrow x - a > 0 \text{ and } b - x > 0.$$

By putting $\varepsilon = b - x$ in (1) and $\delta = x - a$ in (2) we get $x < b'$ and $a' < x$, since $a', b' \in I$, then:

$$a' < x \leq b' \Rightarrow x \in I \Rightarrow]a, b[\subset I.$$

So

$$I =]a, b[.$$

In the same way we prove that I *is a interval in the other cases.*