Chapter two: Complex numbers

2.1 Definitions and properties

Definition 2.1

Any number which can be expressed in the form $x + yi$ where x, y are real numbers and $i^2 = -1$, is called a complex number.

A complex number is, generally, denoted by the letter z. i.e. $z = x + yi$, 'x' is called the real part of z and is written as $Re\, z$ and 'y' is called the imaginary part of z and is written as $Im\, z$.

If $x = 0$ and $y \neq 0$, then the complex number becomes yi which is a purely imaginary complex

If $y = 0$ then the complex number becomes 'x' which is a real number.

The set of complex numbers, denoted by $\mathbb{C}$.

Definition 2.2 (Algebra of complex numbers **)**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

(a) The complex numbers z_1 and z_2 are said to be equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
(b) $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$.
(c) $z_1 . z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$.
(d) For any non-zero complex number $z = x + i\, y$, there exists a multiplicative inverse denoted $\frac{1}{z}$ where $\frac{1}{z} = \frac{1}{x + i\, y} = \frac{x - i\, y}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\, \frac{-y}{x^2 + y^2}$.
Note The set of complex numbers equipped with the operations of addition and multiplication is a commutative filed.

Definition 2.3 (Conjugate of a complex number **)**

Let $z = x + iy$ be a complex number. The complex number $x - iy$ is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = x - iy$.
Properties

Let z, z_1 and z_2 be a complex numbers. We have :
1. $\overline{(\bar{z})} = z$.
2. $z + \bar{z} = 2\, Re\, (z)\, , z - \bar{z} = 2\, i\, Im(z)$.
3. $z = \bar{z} \Leftrightarrow z$ is real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.
5. $z\, .\bar{z} = \{Re\, (z)\}^2 + \{Im\, (z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z_1} + \bar{z_2}$.
7. $\overline{(z_1\, .z_2)} = \bar{z_1}\, .\bar{z_2}$.
8. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$ $(z_2 \neq 0)$.

Definition 2.4 (Modulus of a complex number **)**

Let $z = x + iy$ be a complex number. Then the positive real number $\sqrt{x^2 + y^2}$ is called modulus (absolute value) of z and it is denoted by $|z|$ i.e., $|z| = \sqrt{x^2 + y^2}$.
Properties

Let z, z_1 and z_2 be complex numbers. We have :

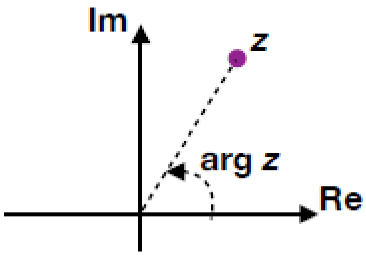
1. $|Re\,(z)| \leq |z|$.
2. $|Im\,(z)| \leq |z|$.
3. $|z| = 0 \Leftrightarrow z = 0$.
4. $z\,.\bar{z} = |z|^2$.
5. $|z_1 z_2| = |z_1||z_2|$.
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.
7. $|z_1 + z_2| = |z_1| + |z_2|$.

Definition 2.5 (Argument of the complex number **)**

The complex number z is represented by point P, we can join point P to the origin with a line segment.

If $z \neq 0$ the angle from the positive axis to the line segment is called the argument of the complex number z.

The argument of z is denoted by $\text{Arg}\, z$.



Properties

Let z, z_1 and z_2 be non zero complex numbers We have :

1. $\text{Arg}\, \bar{z} = -\text{Arg}\, z + 2\pi k$, where $k \in \mathbb{Z}$.
2. $\text{Arg}\,(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2 + 2\pi k$, where $k \in \mathbb{Z}$.
3. $\text{Arg}\, \frac{z_1}{z_2} = \text{Arg}\, z_1 - \text{Arg}\, z_2 + 2\pi k$, where $k \in \mathbb{Z}$.

2.2.The trigonometric form and exponential form a complex number

Definition 2.6

The trigonometric form of a complex number $z = x + yi$ is $z = r(\cos\theta + i\sin\theta)$, where r is the modulus of z, and θ is the argument of z.

Putting $e^{i\theta} = \cos\theta + i\sin\theta$ and since $z = r(\cos\theta + i\sin\theta)$ we therefore obtain another way in which to denote a complex number: $z = re^{i\theta}$, called the exponential form.

So

$$z = \underbrace{x + yi}_{\text{Algebraic form}} = \underbrace{re^{i\theta}}_{\text{Exponentiel form}} = \underbrace{r(\cos\theta + i\sin\theta)}_{\text{Trigonometric form}}.$$

where $r = \sqrt{x^2 + y^2}$; $\cos\theta = \frac{y}{r}$; $i\sin\,\frac{y}{r}$.

Remark Let $z \in \mathbb{C}^*$ where $z = x + iy$, so

$$\text{Arg}(z)=\begin{cases}\dfrac{\pi}{2} & if\ x=0; y>0\\ -\dfrac{\pi}{2} & if\ x=0; y<0\\ \arctan\dfrac{y}{x} & if\ x>0\\ \arctan\dfrac{y}{x}+\pi & if\ x<0\end{cases}$$

Properties

Let z, z_1 and z_2 be non zero complex numbers where $z = re^{i\theta}$, $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$.
We have :

1. $\bar{z} = re^{-i\theta}$.
2. $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$.
3. $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.
4. $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$.
5. $\forall\, n \in \mathbb{Z}$: $z^n = r^ne^{in\theta}$. (De moivre's formula).
6. $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$; $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. (Euler formula).

2.3 Application of complex numbers to trigonometry

2.3.1 Calculates $\cos nx$ and $\sin nx$ based en $\cos x$ and $\sin x$

We have:

$$\begin{aligned}\cos nx + i\sin nx &= (\cos x + i\sin x)^n\\ &= \sum_{k=0}^{n} C_n^k i^k \cos^{n-k} x \sin^k x\\ &= C_n^0\cos^n x - C_n^2\cos^{n-2} x \sin^2 x + C_n^4\cos^{n-4} x \sin^4 x - \cdots \dots\dots\dots\\ &+i(C_n^1\cos^{n-1} x\ \sin x - C_n^3\cos^{n-3} x \sin^3 x + C_n^5\cos^{n-5} x \sin^5 x + \cdots \dots).\end{aligned}$$

So

$$\begin{cases}\cos nx = C_n^0\cos^n x - C_n^2\cos^{n-2} x \sin^2 x + C_n^4\cos^{n-4} x \sin^4 x - \cdots \dots\dots\dots\\ \sin nx = C_n^1\cos^{n-1} x\ \sin x - C_n^3\cos^{n-3} x \sin^3 x + C_n^5\cos^{n-5} x \sin^5 x - \cdots \dots\end{cases}.$$

Or

$$\begin{cases}\cos nx = \displaystyle\sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k C_n^{2k} \cos^{n-2k} x \sin^{2k} x\\ \sin nx = \displaystyle\sum_{k=0}^{E\left(\frac{n+1}{2}\right)} (-1)^{k-1} C_n^{2k-1} \cos^{n-2k+1} x \sin^{2k-1} x\end{cases}.$$

Where $E\left(\frac{n}{2}\right)$ denotes the integer part of the rational number $\frac{n}{2}$.

2.3.2 Linearization of $\cos^n x$ and $\sin^n x$.

For obtain linearization of $\cos^n x$ and $\sin^n x$ we use the relations.

$$\forall\,\theta \in \mathbb{R}: \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\ ;\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

and

$$\forall\, k \in \mathbb{Z};\ \forall\, x \in \mathbb{R}: e^{ikx} + e^{-ikx} = 2\cos kx;\ e^{ikx} - e^{-ikx} = 2i\sin kx.$$

Example

Write in the linear form $\cos^3 x$ and $\sin^3 x$.
We have:

$$\cos^3 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3$$
$$= \frac{1}{8}\left(e^{3ix}+e^{-3ix}+3\left(e^{ix}+e^{-ix}\right)\right)$$
$$= \frac{1}{8}\left(2\cos 3x + 3(2\sin x)\right)$$
$$= \frac{1}{4}\cos 3x + \frac{3}{4}\sin x.$$
$$\sin^3 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3$$
$$= \frac{e^{3ix}-e^{-3ix}-3\left(e^{ix}-e^{-ix}\right)}{-8i}$$
$$= \frac{2i\sin 3x - 3(2i\sin x)}{-8i}$$
$$= -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x.$$

2.3.3 $\mathbf{n}^{th}$ roots of complex number

Definition 2.7

Let $n \in \mathbb{N}^* - \{1\}$ An nth root of complex number a is a complex number z such that $z^n = a$.

Theorem 2.1

Any nonzero complex number has exactly $n \in \mathbb{N}$ distinct nth roots. The roots lie on a circle of radius $|z|$ centred at the origin and spaced out evenly by angles of $\frac{2\pi}{n}$. Concretely, if $a = re^{i\theta}$, then solutions to $z^n = a$ are given by $z_k = \sqrt[n]{r}\, e^{i\frac{\theta+2\pi k}{n}}$ for $k \in \{0,1,\dots,n-1\}$.

Proof

Assume that $a = re^{i\theta}$ and $z = \rho e^{i\alpha}$, so

$$z^n = a \Leftrightarrow \rho^n e^{in\alpha} = re^{i\theta}$$
$$\Leftrightarrow \rho e^{in\alpha} = re^{i\theta}$$
$$\Leftrightarrow \begin{cases} \rho^n = r \\ \text{and} \\ n\alpha = \theta + 2\pi k, k \in \mathbb{Z} \end{cases}$$
$$\Leftrightarrow \begin{cases} \rho = \sqrt[n]{r} \\ \text{and} \\ \alpha = \dfrac{\theta + 2\pi k}{n}, k \in \mathbb{Z}. \end{cases}$$

The expression for z takes n different values for $k = 0;\ 1; \dots\dots;\ n-1$, and the values start to repeat for $k = n, n+1, \dots\dots$

Hence the expression for the n nth roots of a: $z_k = \sqrt[n]{r}\, e^{i\frac{\theta+2\pi k}{n}}$ for $k \in \{0,1,\dots,n-1\}$.

Examples

1) The n nth roots of unity are therefore the numbers $z_k = e^{i\frac{\theta+2\pi k}{n}} = \cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}$ for $k \in \{0,1,\dots,n-1\}$.

2) Solve in $\mathbb{C}$ the equation $z^7 = \bar{z}$.

Answer

one of solutions is obviously $z = 0$. For other solutions the simple way is to write $z = re^{i\theta}$, then

$$z^7 = \bar{z} \Leftrightarrow r^n e^{in\theta} = re^{-i\theta}$$

$$\Leftrightarrow \begin{cases} r^7 = r \\ \quad \text{and} \\ 7\theta = -\theta + 2\pi k, k \in \mathbb{Z}, \end{cases}$$

$$\Leftrightarrow \begin{cases} r(r^6 - 1) = 0 \\ \quad \text{and} \\ \theta = \dfrac{2\pi k}{8}, k \in \mathbb{Z}, \end{cases}$$

$$\Leftrightarrow \begin{cases} r = 1 \\ \quad \text{and} \\ \theta = \dfrac{\pi k}{4}, \text{for } k \in \{0,1,2,3,4,5,6,7\}. \end{cases}$$

So the set of solutions is $S = \left\{0, e^{i\frac{\pi k}{4}} \text{ for } k \in \{0,1,2,3,4,5,6,7\}\right\}$.

Exercises

1) Write $\cos^5 x$ in linear form.

2) a) Use the De Mover's formula to prove that: $\sin 5\theta = \sin\theta\,(16\cos^4\theta - 12\cos^2\theta - 1)$

b) Solve the equation $16\,x^4 - 12\,x^2 - 1$and determine the value of $\cos\frac{\pi}{5}$.

3) The following finite sum S, are given by $S = 1 + \cos\theta + \cos 2\theta + \cdots \ldots + \cos(n-1)\theta$, where $\theta \neq 2\pi k, k \in \mathbb{Z}$ and $n \in \mathbb{N}^*$.

Using the demoivre's formula prove that: $S = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} + \frac{1}{2}$.

4)* One of the roots of the equation $z^7 - 1 = 0$ is denoted by ω , where $0 < \arg\omega < \frac{\pi}{3}$.

a) Find ω in the form $re^{i\theta}, r > 0, 0 < \theta < \frac{\pi}{3}$.

b) Show clearly that $1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

c) Using the results of the previous parts, deduce that: $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

5)* Calculate: $S = \sum_{p=0}^{n-1} \frac{\sin px}{\cos^p x}$ (Using the sum $\sum_{p=0}^{n-1} \frac{e^{ipx}}{\cos^p x}$).

Solutions

1) $\forall k \in \mathbb{Z}; \forall x \in \mathbb{R} : \cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$

$\cos^5 x = \left(\dfrac{e^{ix} + e^{-ix}}{2}\right)^5$ the Pascal's triangle for $n = 5$ is

$n=0$	1					
$n=1$	1	1				
$n=2$	1	2	1			
$n=3$	1	3	3	1		
$n=4$	1	4	6	4	1	
$n=5$	1	5	10	10	5	1

so

$$\begin{aligned} \cos^5 x &= \frac{1}{32}\left(e^{i5x} + 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} + 10e^{2ix}e^{-3ix} + 5e^{ix}e^{-4ix} + e^{-i5x}\right) \\ &= \frac{1}{32}\left[\left(e^{i5x} + e^{-i5x}\right) + 5\left(e^{i3x} + e^{-i3x}\right) + 10\left(e^{ix} + e^{-ix}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{32}2\left(\cos 5x + 5\cos 3x + 10\cos x\right)\\
&= \frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x.
\end{aligned}
$$

2) a) we have:

$$\sin n\theta = \sum_{i=1}^{\left[\frac{n+1}{2}\right]} (-1)^{i-1} C_n^{2i-1} \cos^{n-2i+1}\theta \sin^{2i-1}\theta$$

so

$$
\begin{aligned}
\sin 5\theta &= \sum_{i=1}^{\left[\frac{5+1}{2}\right]} (-1)^{i-1} C_5^{2i-1} \cos^{5-2i+1}\theta \sin^{2i-1}\theta = \sum_{i=1}^{3} (-1)^{i-1} C_5^{2i-1} \cos^{6-2i}\theta \sin^{2i-1}\theta\\
&= C_5^1 \cos^4\theta \sin\theta - C_5^3 \cos^2\theta \sin^3\theta + C_5^5 \sin^5\theta\\
&= 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta\\
&= 5\cos^4\theta \sin\theta - 10\cos^2\theta \left(1-\cos^2\theta\right)\sin\theta + \left(1-\cos^2\theta\right)^2 \sin\theta
\end{aligned}
$$

so

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta - 1\right) \text{........................} (1)$$

b) $16x^4 - 12x^2 + 1 = 0$, Solutions is: $\frac{1}{4}\sqrt{5}+\frac{1}{4}, \frac{1}{4}-\frac{1}{4}\sqrt{5}, \frac{1}{4}\sqrt{5}-\frac{1}{4}, -\frac{1}{4}\sqrt{5}-\frac{1}{4}$.
Let us take $\theta = \frac{\pi}{5}$ in equality (1) we obtain:

$$\sin\frac{\pi}{5}\left(16\cos^4\frac{\pi}{5} - 12\cos^2\frac{\pi}{5} - 1\right) = 0$$

so

$$16\cos^4\frac{\pi}{5} - 12\cos^2\frac{\pi}{5} - 1 = 0$$

From the last equality, we conclude that $\cos\frac{\pi}{5}$ is one of the solutions to the previous equation, so

$\cos\frac{\pi}{5} = \frac{1}{4}\sqrt{5} - \frac{1}{4}$ (because $0 < \frac{\pi}{5} < \frac{\pi}{3} \Longrightarrow \cos\frac{\pi}{5} \in \left]\frac{1}{2}, 1\right[$).

3) S is the imaginary part of $A = 1 + e^{i\theta} + e^{2i\theta} + \text{..........} e^{i(n-1)\theta}$

so

$$
\begin{aligned}
A &= 1 + e^{i\theta} + e^{2i\theta} + \text{..........} e^{i(n-1)\theta}\\
&= 1 + e^{i\theta} + \left(e^{i\theta}\right)^2 + \text{..........} \left(e^{i\theta}\right)^{(n-1)}\\
&= \frac{\left(e^{i\theta}\right)^n - 1}{e^{i\theta} - 1} = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}\\
&= \frac{e^{in\theta} - 1}{e^{i\theta} - 1}\frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}} = \frac{e^{i\left(n-\frac{1}{2}\right)\theta} - e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}\\
&= \frac{\cos\left(n-\frac{1}{2}\right)\theta - \cos\frac{\theta}{2} + i\left(\sin\left(n-\frac{1}{2}\right)\theta + \sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}.
\end{aligned}
$$

We obtain:

$$S = \operatorname{Im}(A) = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + \frac{1}{2}.$$