Chapter three: Real sequences

3 Real sequences
3.1 Generalities
Definition 3.1
We call each function U of $\mathbb{N}$ in $\mathbb{R}$; a real sequence.

$$U : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longrightarrow U(n) = U_n$$

. U_n is called the general term of the sequence U.
. We also symbolize the sequence by (U_n) or $(U_n)_{n\in\mathbb{N}}$ or $(U_n)_{n\geq n_0}$ if the sequence is defined for each $n \geq n_0$.
. A real sequence is defined explicitly or with a recurrent relation.

Examples 3.1
1) $(u_n)_{n\geq 2}$ is a sequence defined by its general term:

$$\forall n \geq 2: u_n = \sqrt{n-2}.$$

we have

$$u_2 = 0;\ u_3 = 1;\ u_4 = \sqrt{2};\ u_5 = \sqrt{3} \dots\dots\dots\dots; u_{12} = \sqrt{10}; \dots\dots$$

2) $(v_n)_{n\in\mathbb{N}}$ is a sequence defined by the following recurrent relation:

$$u_0 = 1;\ \forall n \in \mathbb{N}: u_{n+1} = \frac{u_n}{u_n + 1}$$

So

$$u_0 = 1;\ u_1 = \frac{u_0}{u_0+1} = \frac{1}{2};\ u_2 = \frac{u_1}{u_1+1} = \frac{1}{3};\ u_3 = \frac{u_2}{u_2+1} = \frac{1}{4}; \dots\dots\dots\dots$$

Prove that $\forall n \in \mathbb{N}: u_n = \frac{1}{n+1}$.

Definition 3.2
Let (u_n) be a real sequence.
. (u_n) is bounded from above if and only if :

$$\exists M \in \mathbb{R}; \forall n \in \mathbb{N}: u_n \leq M.$$

. (u_n) is bounded from below if and only if :

$$\exists m \in \mathbb{R}; \forall n \in \mathbb{N}: u_n \geq m.$$

. (u_n) is bounded if and only if it is bounded from above and from below, in other words:

$$\big((u_n)\text{is bounded}\big) \iff \exists M \in \mathbb{R}_+^*; \forall n \in \mathbb{N}: |u_n| \leq M.$$

Example 3.2
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence defined by: $\forall n \in \mathbb{N}: u_n = \frac{n}{2n+1}$.

We have $\forall n \in \mathbb{N}: u_n = \frac{n}{2n+1} = \frac{1}{2} - \frac{1}{2}\frac{1}{2n+1}$, so $\forall n \in \mathbb{N}$:

$$\begin{aligned} n \geq 0 &\Rightarrow 2n + 1 \geq 1 \\ &\Rightarrow 0 > -\frac{1}{2}\frac{1}{2n+1} \geq -\frac{1}{2} \\ &\Rightarrow \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\frac{1}{2n+1} \geq 0 \\ &\Rightarrow \frac{1}{2} > u_n \geq 0. \end{aligned}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded.

Definition 3.3

Let (u_n) be a real sequence.

. (u_n) is increasing (strictly increasing, respectively) if and only if :

$$\forall n \in \mathbb{N}: u_n \leq u_{n+1} \;\; (u_n < u_{n+1}, \text{respectively}).$$

. (u_n) is decreasing (strictly decreasing, respectively) if and only if :

$$\forall n \in \mathbb{N}: u_n \geq u_{n+1} \;\; (u_n > u_{n+1}, \text{respectively}).$$

. (u_n) is constant if and only if :

$$\forall n \in \mathbb{N}: u_n = u_{n+1}.$$

A sequence of real numbers (u_n) is said to be monotonic if it is either increasing or decreasing.

Example 3.3

The sequence $(u_n)_{n\in\mathbb{N}}$, defined in the previous example, is increasing. Indeed

$$\begin{aligned}\forall n \in \mathbb{N}: u_{n+1} - u_n &= \frac{n+1}{2n+3} - \frac{n}{2n+1} \\ &= \frac{1}{(2n+3)(2n+1)} \\ &\geq 0.\end{aligned}$$

3.2 Convergent sequences

Definition 3.4

A sequence (u_n) is convergent and its limit is the real number ℓ if and only if:

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: (n > N \Longrightarrow \; |u_n - \ell| < \varepsilon).$$

And we write $\lim\limits_{n\to\infty} u_n = \ell$ or $\lim u_n = \ell$.

Example 3.4

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence defined by: $\forall n \in \mathbb{N}: u_n = \frac{n}{2n+1}$.

Let's prove that $\lim u_n = \frac{1}{2}$.

Let $\varepsilon > 0$ where $\left|u_n - \frac{1}{2}\right| < \varepsilon$, so

$$\begin{aligned}\left|u_n - \frac{1}{2}\right| < \varepsilon &\Leftrightarrow \left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon \\ &\Leftrightarrow \frac{1}{4n+2} < \varepsilon \\ &\Leftrightarrow n > \frac{1}{4\varepsilon} - \frac{1}{2}.\end{aligned}$$

We have $\frac{1}{4\varepsilon} - \frac{1}{2} \leq \left|\frac{1}{4\varepsilon} - \frac{1}{2}\right| < E\left(\left|\frac{1}{4\varepsilon} - \frac{1}{2}\right|\right) + 1$, so it is enough to take $N = E\left(\left|\frac{1}{4\varepsilon} - \frac{1}{2}\right|\right) + 1$.

Remark 3.1

We can be determine the number N in another way. According to Archimedean axiom there exists $N_0 \in \mathbb{N}$, where $N_0 > \frac{1}{4\varepsilon} - \frac{1}{2}$, so it is enough to chose $N = N_0$.

Theorem 3.1 (Uniqueness of limit)

Every convergent sequence has a unique limit.

Proof

Assume that the sequence (u_n) has two different limits ℓ and ℓ' ($\ell' \neq \ell$), taking $\varepsilon = \frac{|\ell'-\ell|}{2}$.
which implies $\begin{cases} \exists N_0 \in \mathbb{N}; \forall n \in \mathbb{N}: n > N_0 \Longrightarrow\ |u_n - \ell| < \varepsilon \\ \exists N_1 \in \mathbb{N}; \forall n \in \mathbb{N}: n > N_1 \Longrightarrow\ |u_n - \ell'| < \varepsilon. \end{cases}$
Putting $N = \max\{N_0, N_1\}$, then

$$\begin{aligned} \forall n \in \mathbb{N}: n > N \Longrightarrow |\ell' - \ell| &= |u_n - \ell - (u_n - \ell')| \\ &\leq |u_n - \ell| + |u_n - \ell'| \\ &< 2\varepsilon = |\ell' - \ell|. \\ \Longrightarrow |\ell' - \ell| &< |\ell' - \ell|, \text{it's a contradiction.} \end{aligned}$$

Theorem 3.2
If (u_n) is a convergent sequence, then it is a bounded sequence.
Proof

We assume that the sequence (u_n) is convergent to the number ℓ, then for $\varepsilon = 1$ we have:

$$\begin{aligned} \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: n > N \Longrightarrow\ &|u_n - \ell| < 1 \\ &\Longrightarrow \ell - 1\ < u_n < \ell + 1. \end{aligned}$$

Putting $A = \{u_0, u_1, \ldots\ldots, u_N, \ell - 1, \ell + 1\}$, then ; $\forall n \in \mathbb{N}:\ \min A\ \leq u_n \leq \max A$.

Theorem 3.3
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence
1. If (u_n) is increasing and bounded from above, then (u_n) converges, and we have $\lim u_n = \sup_{n\in\mathbb{N}} u_n$.
2. If (u_n) is decreasing and bounded from below, then (u_n) converges, and we have $\lim u_n = \inf_{n\in\mathbb{N}} u_n$.
Remark 3.2 Every bounded monotonic sequence is a convergent sequence.
Proof
1. Let the sequence (u_n) is increasing and bounded from above, then the set $A = \{u_n, n \in \mathbb{N}\}$ is bounded from above putting $\sup A = \ell$.

We have $\begin{cases} \forall n \in \mathbb{N}: u_n \leq \ell \\ \forall \varepsilon > 0; \exists N \in \mathbb{N}: \ell - \varepsilon < u_N. \end{cases}$
On the other hand, since (u_n) is increasing we have:

$$\begin{aligned} \forall n \in \mathbb{N}: n > N &\Longrightarrow u_n \geq u_N \\ &\Longrightarrow \ell \geq u_n \geq u_N > \ell - \varepsilon \\ &\Longrightarrow \ell + \varepsilon > u_n > \ell - \varepsilon \\ &\Longrightarrow |u_n - \ell| < \varepsilon. \end{aligned}$$

Hence

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: (n > N \Longrightarrow\ |u_n - \ell| < \varepsilon).$$

2. Let the sequence (u_n) is decreasing and bounded from below, then the set $A = \{u_n, n \in \mathbb{N}\}$ is bounded from below putting $\inf A = \ell'$.

We have $\begin{cases} \forall n \in \mathbb{N}: u_n \geq \ell' \\ \forall \varepsilon > 0; \exists N \in \mathbb{N}: \ell' + \varepsilon > u_N. \end{cases}$

On the other hand, since (u_n) is decreasing we have:

$$\begin{aligned} \forall n \in \mathbb{N}: n > N &\Longrightarrow u_n \leq u_N \\ &\Longrightarrow \ell' \leq u_n \leq u_N < \ell' + \varepsilon \\ &\Longrightarrow \ell' - \varepsilon < u_n < \ell' + \varepsilon \\ &\Longrightarrow |u_n - \ell'| < \varepsilon. \end{aligned}$$

Hence

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: (n > N \Longrightarrow |u_n - \ell'| < \varepsilon).$$

Example 3.5

Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined as follows:

$$\forall n \in \mathbb{N}: u_{n+1} = \frac{2u_n + 1}{u_n + 2} \quad \text{and} \quad u_0 = \alpha > 1$$

First we will show that the sequence (u_n) is bounded from below by 1. We prove by induction that $\forall n \in \mathbb{N}: u_n > 1$.

Since $u_0 = \alpha > 1$, it is true.

Next suppose $u_n > 1$, and we have $u_{n+1} = \frac{2u_n+4-4+1}{u_n+2} = 2 - \frac{3}{u_n+2}$, then

$$\begin{aligned} u_n > 1 &\Longrightarrow \frac{3}{u_n + 2} < 1 \\ &\Longrightarrow 2 - \frac{3}{u_n + 2} > 1 \\ &\Longrightarrow u_{n+1} > 1. \end{aligned}$$

Next we show that the sequence (u_n) is decreasing, indeed,

$$\begin{aligned} \forall n \in \mathbb{N}: u_{n+1} - u_n &= \frac{2u_n + 1}{u_n + 2} - u_n \\ &= \frac{1 - u_n^2}{u_n + 2} \\ &= \frac{(1 + u_n)(1 - u_n)}{u_n + 2} \\ &< 0 \ \text{(Since } u_n > 1 \text{).} \end{aligned}$$

Thus (u_n) is an decreasing sequence that is bounded from below.

By the monotone convergence theorem (u_n), converges.

Theorem 3.4

If the sequences (u_n) and (v_n) are converges towards ℓ and ℓ' respectively then the sequences $(u_n + v_n)$, $(u_n v_n)$, (λu_n) and $(|u_n|)$ are converges towards $\ell + \ell'$, $\ell\ell'$, $\lambda\ell$, $|\ell|$ respectively. Also if $\ell' \neq 0$ and $\forall n \in \mathbb{N}: u_n \neq 0$ then the sequence $\left(\frac{u_n}{v_n}\right)$ converges towards $\frac{\ell}{\ell'}$.

Proof (Let us prove the last case)

We have $\lim v_n = \ell' \neq 0$ taking $\varepsilon = \frac{|\ell'|}{2}$, which implies ; $\exists N_0 \in \mathbb{N}; \forall n \in \mathbb{N}$:

$$\begin{aligned} n > N_0 &\Longrightarrow |v_n - \ell'| < \varepsilon \\ &\Longrightarrow \big||v_n| - |\ell'|\big| < \frac{|\ell'|}{2} \\ &\Longrightarrow \frac{|\ell'|}{2} < |v_n| < \frac{3|\ell'|}{2} \\ &\Longrightarrow \frac{2}{3|\ell'|} < \frac{1}{|v_n|} < \frac{2}{|\ell'|}. \end{aligned}$$

So

$$n > N_0 \Longrightarrow \frac{1}{|v_n|} < \frac{2}{|\ell'|}.$$

On the other hand for $\varepsilon > 0$, then: $\begin{cases} \exists N_1 \in \mathbb{N}; \forall n \in \mathbb{N}: n > N_1 \Longrightarrow |u_n - \ell| < \varepsilon \\ \exists N_2 \in \mathbb{N}; \forall n \in \mathbb{N}: n > N_2 \Longrightarrow |v_n - \ell'| < \varepsilon. \end{cases}$

Putting $N = \max\{N_0, N_1, N_2\}$, then $\forall n \in \mathbb{N}$:

$$\begin{aligned} n > N \Longrightarrow \left|\frac{u_n}{v_n} - \frac{\ell}{\ell'}\right| &= \left|\frac{\ell' u_n - \ell v_n}{\ell' v_n}\right| \\ &= \left|\frac{\ell'(u_n - \ell) - \ell(v_n - \ell')}{\ell' v_n}\right| \\ &\leq \frac{|\ell'||(u_n - \ell)| + |\ell||(v_n - \ell')|}{|\ell'||v_n|} \\ &< \frac{2\varepsilon(|\ell'| + |\ell|)}{|\ell'|^2} = \varepsilon_0. \end{aligned}$$

So

$$\forall \varepsilon_0 > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: \left(n > N \Longrightarrow \left|\frac{u_n}{v_n} - \frac{\ell}{\ell'}\right| < \varepsilon_0\right).$$

Theorem 3.5

1. Let (u_n) and (v_n) two sequences that converges towards ℓ and ℓ' respectively, where $\forall n \in \mathbb{N}: u_n \leq v_n$ (or $u_n < v_n$) then $\ell \leq \ell'$.
2. (Squeeze Theorem) Let (u_n), (v_n) and (w_n) three sequences such that $\forall n \in \mathbb{N}: v_n \leq u_n \leq w_n$ (or $v_n < u_n < w_n$) then:

$$\left(\lim v_n = \lim w_n = \ell\right) \Longrightarrow \lim u_n = \ell.$$

Proof

1. (prove by contradiction)

Assume that $\ell > \ell'$, take $\varepsilon = \frac{\ell - \ell'}{2}$ which implies ; $\exists N_0, N_1 \in \mathbb{N};\ \forall n \in \mathbb{N}$:

$$\begin{cases} n > N_0 \Longrightarrow |u_n - \ell| < \varepsilon = \dfrac{\ell - \ell'}{2} \Longrightarrow \dfrac{\ell + \ell'}{2} < u_n < \dfrac{3\ell - \ell'}{2} \\ n > N_1 \Longrightarrow |v_n - \ell'| < \varepsilon = \dfrac{\ell - \ell'}{2} \Longrightarrow \dfrac{3\ell' - \ell}{2} < v_n < \dfrac{\ell + \ell'}{2}. \end{cases}$$

For $N = \max\{N_0, N_1\}$ then $\forall n \in \mathbb{N}: n > N \Longrightarrow \ v_n < \frac{\ell+\ell'}{2} < u_n$. and this contradicts the hypothesis $\forall n \in \mathbb{N}: u_n \leq v_n$.
2. We have $\lim v_n = \lim w_n = \ell \Leftrightarrow \forall \varepsilon > 0; \exists N_0, N_1 \in \mathbb{N};\ \forall n \in \mathbb{N}$:

$$\begin{cases} n > N_0 \Longrightarrow \ |v_n - \ell| < \varepsilon \Longrightarrow \ell - \varepsilon \ < v_n < l + \varepsilon \\ n > N_1 \Longrightarrow \ |w_n - \ell| < \varepsilon \Longrightarrow \ell - \varepsilon \ < w_n < l + \varepsilon. \end{cases}$$

For $N = \max\{N_0, N_1\}$ then $\forall n \in \mathbb{N}: n > N \Longrightarrow \ \ell - \varepsilon \ < v_n \ \leq u_n \leq w_n < l + \varepsilon$, so

$$\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}: n > N \Longrightarrow \ |v_n - \ell| < \varepsilon \ .$$

3.3 Subsequences

Definition 3.5

Let (u_n) be a sequence. A subsequence (v_k) of the sequence (u_n) is defined by a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that f is strictly increasing, and $v_k = u_{f(k)}$ for $k \in \mathbb{N}$.

We often write n_k instead of $f(k)$.

Example 3.6

Let (u_n) be a sequence defined by $\forall n \in \mathbb{N}: u_n = \frac{n}{n+1}$.

For $n_k = f(k) = 3k$ (f is strictly increasing) the subsequence (v_k) (or (u_{n_k})) is defined by: $\forall k \in \mathbb{N}: v_k = u_{3k} = \frac{3k}{3k+1}$.

For $n'_k = g(k) = k^2$ (g is strictly increasing) the subsequence (w_k) (or $(u_{n'_k})$) is defined by: $\forall k \in \mathbb{N}: w_k = u_{k^2} = \frac{k^2}{k^2+1}$.

The following table shows the relationship of the subsequences (v_k) and (w_k) to the sequence (u_n).

u_n	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	....
	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$	....
$v_k = u_{3k}$	v_0			v_1			v_2			v_3			....
	0			$\frac{3}{4}$			$\frac{6}{7}$			$\frac{9}{10}$			....
$w_k = u_{k^2}$	w_0	w_1			w_2					w_3			....
	0	$\frac{1}{2}$			$\frac{4}{5}$					$\frac{9}{10}$			....

Proposition 3.1

If (n_k) is a sequence of strictly increasing natural numbers, then $\forall k \in \mathbb{N}: n_k \geq k$.

Proof (By induction **)**

For $k = 0$ we have $n_0 \geq 0$ (It is true because $n_0 \in \mathbb{N}$).

Assume that $\forall k \in \mathbb{N}: n_k \geq k$. Since (nk) is strictly increasing, then

$$\begin{aligned} n_{k+1} > n_k \Longrightarrow n_{k+1} &> k \\ \Longrightarrow n_{k+1} &\geq k + 1. \end{aligned}$$

Theorem 3.6

If a sequence is convergent, then any sub-sequence of it converges to the same limit

Proof

Let (u_n) be a convergent sequence towards ℓ and let n_k be a sequence of strictly increasing natural numbers, we constructing the subsequence (v_k) that is defined by $\forall k \in \mathbb{N}: v_k = u_{n_k}$ and let's prove that: $\lim\limits_{k\to\infty} v_k = \ell$.

We have $\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}: n > N \Longrightarrow \ |u_n - \ell| < \varepsilon$.

On other hand, since (n_k) is strictly increasing then:

$$\forall k \in \mathbb{N}: k > N \Longrightarrow n_k > n_N$$
$$\Longrightarrow n_k > n_N \geq N \text{ (According to proposition 3.1)}$$
$$\Longrightarrow n_k > N$$
$$\Longrightarrow |u_{n_k} - \ell| < \varepsilon$$
$$\Longrightarrow |v_k - \ell| < \varepsilon.$$

So

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall k \in \mathbb{N}: k > N \Longrightarrow |v_k - \ell| < \varepsilon$$

We conclude that the subsequence (v_k) is converges towards ℓ.

Remark 3.3

To prove the divergence of certain sequences, we can use the contrapositive implication in theorem (3.6).

Example.3.7

Let the sequence (u_n) be defined by $\forall n \in \mathbb{N}: u_n = \frac{n(-1)^n}{n+1}$, we will show that the sequence (u_n) is divergent.

We constructing the two subsequences (u_{2k}) and(u_{2k+1}), where $\forall k \in \mathbb{N}$:

$$\begin{cases} u_{2k} = \dfrac{2k(-1)^{2k}}{2k+1} = \dfrac{2k}{2k+1} \\ u_{2k+1} = \dfrac{(2k+1)(-1)^{2k+1}}{2k+2} = -\dfrac{(2k+1)}{2k+2}. \end{cases}$$

We have

$$\lim_{k\to\infty} u_{2k} = \lim_{k\to\infty} \frac{2k}{2k+1} = 1,$$
$$\lim_{k\to\infty} u_{2k+1} = \lim_{k\to\infty} -\frac{2k+1}{2k+2} = -1.$$

Since $\lim\limits_{k\to\infty} u_{2k} \neq \lim\limits_{k\to\infty} u_{2k+1}$ then the sequence (u_n) is divergent.

Definition 3.6

A sequence (u_n) diverges to infinity if and only if

$$\forall A \in \mathbb{R}; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: n > N \Longrightarrow u_n > A.$$

In this case we write $\lim\limits_{n\to\infty} u_n = +\infty$.

Similarly, a sequence (u_n) diverges to minus infinity and we write $\lim\limits_{n\to\infty} u_n = -\infty$, if and only if:

$$\forall A \in \mathbb{R}; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: n > N \Longrightarrow u_n < -A.$$

Proposition 3.2

If (u_n) is an increasing and unbounded sequence from above. Then

$$\lim_{n\to\infty} u_n = +\infty$$

If (u_n) is an decreasing and unbounded sequence from below. Then

$$\lim_{n\to\infty} u_n = -\infty$$

Proof

Assume that (u_n) is an increasing and unbounded sequence from above.

Since (u_n) is unbounded from above then $\forall A \in \mathbb{R}; \exists N \in \mathbb{N}: u_N > A$.

And Since (u_n) is increasing we have $\forall n \in \mathbb{N}: n > N \Longrightarrow u_n \geq u_N$

$$\Longrightarrow u_n > A$$

So

$$\forall A \in \mathbb{R}; \exists N \in \mathbb{N}; \forall n \in \mathbb{N}: n > N \Longrightarrow u_n > A.$$

In the same way, we prove the second case.

3.5 Adjacent sequences

Definition 3.7
Two sequences (u_n) and (v_n), are said to be adjacent if and only if one of the two sequences is increasing and the other decreasing and $\lim_{n\to\infty}(u_n - v_n) = 0$.

Theorem 3.7
Every two adjacent sequences are convergent sequences and have the same limit.

Proof
Let (u_n) and (v_n) be two adjacent sequences, where (u_n) is increasing and (v_n) is decreasing, so the sequence $(v_n - u_n)$ is decreasing and converges to 0, this means that $\inf_{n\in\mathbb{N}}(v_n - u_n) = 0$, thus implies

$$\forall n \in \mathbb{N}: v_n - u_n \geq 0 \Longrightarrow v_n \geq u_n$$
$$\Longrightarrow v_0 \geq v_n \geq u_n \geq u_0.$$

So, the sequences (u_n) and (v_n) are monotonic and bounded sequences, and therefore they are convergent.
Next assume that $\lim_{n\to\infty} u_n = \ell$ and $\lim_{n\to\infty} v_n = \ell'$ according to theorem 3.4 we have $\lim_{n\to\infty}(u_n - v_n) = \ell - \ell'$ and in other hand we have $\lim_{n\to\infty}(u_n - v_n) = 0$.So $\ell - \ell' = 0 \Longrightarrow \ell = \ell'$.

Example 3.8
Let (u_n) and (v_n) be two sequences defined by

$$\forall n \in \mathbb{N}: u_n = \sum_{k=1}^{n} \frac{1}{n^2} \text{ and } v_n = \sum_{k=1}^{n} \frac{1}{n^2} + \frac{1}{n}.$$

Let's to show that the sequences (u_n) and (v_n) are adjacent. Indeed, we have

$$\forall n \in \mathbb{N}: u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{n^2} - \sum_{k=1}^{n} \frac{1}{n^2} = \frac{1}{(n+1)^2} > 0.$$

$$v_{n+1} - v_n = \sum_{k=1}^{n+1} \frac{1}{n^2} + \frac{1}{n+1} - \left(\sum_{k=1}^{n} \frac{1}{n^2} + \frac{1}{n}\right)$$
$$= \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{n}$$
$$= -\frac{1}{n(n+1)^2} < 0.$$

So (u_n) is increasing and (v_n) is decreasing.
On the other hand we have $\lim_{n\to\infty}(u_n - v_n) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.
Thus the sequences (u_n) and (v_n) are adjacent.
(It can be proven that $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n = \frac{\pi^2}{6}$.)

Exercise 3.1
1) Let (u_n) be a sequence. Prove that if the two subsequences (u_{2n}) and (u_{2n+1}) converges towards ℓ then the sequence (u_n) converges towards ℓ.

2) Let the sequence (S_n) defined by $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Prove that the two subsequences (S_{2n}) and (S_{2n+1}) are adjacent, what do you conclude ?

Solution

1) Assume that (u_{2n}) and (u_{2n+1}) converge to ℓ then:

$$\forall \varepsilon > 0; \exists N_0, N_1 \in \mathbb{N};\ \forall k \in \mathbb{N}: \begin{cases} k > N_0 \Longrightarrow\ |v_{2k} - \ell| < \varepsilon \\ k > N_1 \Longrightarrow\ |v_{2k+1} - \ell| < \varepsilon. \end{cases}$$

Putting $N = \max\{2N_0, 2N_1 + 1\}$ so $\forall n \in \mathbb{N}$:

If n is pair, then $n = 2k$, so

$$n > N \Longrightarrow\ 2k > 2N_0 \Longrightarrow\ k > N_0 \Longrightarrow\ |u_{2k} - \ell| < \varepsilon \Longrightarrow\ |u_n - \ell| < \varepsilon.$$

If n is odd, then $n = 2k + 1$, so

$$n > N \Longrightarrow\ 2k + 1 > 2N_1 + 1 \Longrightarrow\ k > N_1 \Longrightarrow\ |u_{2k+1} - \ell| < \varepsilon \Longrightarrow\ |u_n - \ell| < \varepsilon.$$

So

$$\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}: n > N \Longrightarrow\ |u_n - \ell| < \varepsilon.$$

2) We have

$$S_{2n+2} - S_{2n} = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \frac{(-1)^{2n+2}}{2n+1} + \frac{(-1)^{2n+3}}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0.$$

$$S_{2n+3} - S_{2n+1} = \sum_{k=1}^{2n+3} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n+1} \frac{(-1)^{k}}{k} = \frac{(-1)^{2n+3}}{2n+2} + \frac{(-1)^{2n+4}}{2n+3} = \frac{-1}{(2n+2)(2n+3)} < 0.$$

So (S_{2n}) is increasing and (S_{2n+1}) is decreasing and

$$\lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} \left(-\frac{1}{2n+1}\right) = 0.$$

So (S_{2n}) and (S_{2n+1}) are adjacent.

Thus the subsequences (S_{2n}) and (S_{2n+1}) are convergent sequences and have the same limit, according to the first question the sequence (S_n) is convergent.

Theorem 3.8 (BOLZANO-WEIERSTRASS**)**

From every bounded real sequence, at least one convergent subsequence can be extracted.

Proof

Let (u_n) be a bounded sequence, we put $a_0 = \inf_{n\in\mathbb{N}} u_n$ and $b_0 = \sup_{n\in\mathbb{N}} u_n$.

We have $\forall n \in \mathbb{N}$: $a_0 \le u_n \le b_0$, we put $I_0 = [a_0, b_0]$.

Let us divide the interval I_0 into two intervals of equal length. At least one of these two intervals contains an infinite number of terms of the sequence (u_n), which we denote by $I_1 = [a_1, b_1]$, and let u_{n_1} be one of the terms of the sequence (u_n), where $u_{n_1} \in I_1$.

Let us divide the interval I_1 into two intervals of equal length. At least one of these two intervals contains an infinite number of terms of the sequence (u_n), which we denote by $I_2 = [a_2, b_2]$, and let u_{n_2} be one of the terms of the sequence (u_n), where $u_{n_2} \in I_2$ and $n_2 > n_1$ (this is possible because I_2 contains an infinite number of terms of the sequence (u_n)).

Thus, we create a sequence of intervals $I_k = [a_k, b_k]$ where I_k is one of the two halves of the interval I_{-1k} which contains an infinite number of terms of the sequence (u_n) and u_{n_k} is one of the terms of the sequence (u_n) where $u_{n_k} \in I_k$ and $n_k > n_{k-1}$, Thus we get a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, satisfies $\forall k \in \mathbb{N}:\ a_k \le u_{n_k} \le b_k$.

We have $\lim_{k\to\infty} u_{n_k} = \lim_{k\to\infty} \frac{b_0 - a_0}{2^k} = 0$.

Since $I_k \subseteq I_{k-1}$, then the sequence $(a_k)_{k\in\mathbb{N}}$ is increasing and the sequence $(b_k)_{k\in\mathbb{N}}$ is decreasing, thus the sequences (a_k) and (b_k) are adjacent, therefore the sequence (u_{n_k}) is convergent and its limit is the common limit of the sequences (a_k) and (b_k).

3.6 Cauchy sequence

Definition 3.8

Let (u_n) be a sequence, we say that (u_n) is a Cauchy sequence if it has the following property:

$$\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall p, q \in \mathbb{N}: (p > N \wedge q > N) \Longrightarrow\ |u_p - u_q| < \varepsilon.$$

Second formula

$$\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n, p \in \mathbb{N}: n > N \Longrightarrow\ |u_{n+p} - u_n| < \varepsilon.$$

Theorem 3.9

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof

Necessary condition

Let (u_n) be a sequence converging towards the real number ℓ, then

$$\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}: n > N \Longrightarrow\ |u_n - \ell| < \frac{\varepsilon}{2}.$$

So

$$\begin{aligned}\forall p, q \in \mathbb{N}: (p > N \wedge q > N) \Longrightarrow\ |u_p - u_q| &= |u_p - \ell - (u_q - \ell)| \\ &\leq |u_p - \ell| + |u_q - \ell| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So, (u_n) is a Cauchy sequence.

Sufficient condition

Since (u_n) is Cauchy sequence, then for $\varepsilon = 1$

$$\exists N_0 \in \mathbb{N};\ \forall n, p \in \mathbb{N}: (n > N_0 \wedge p > N_0) \Longrightarrow\ |u_n - u_p| < 1.$$

For $p = N_0 + 1$, and for every $n > N$ we have $|u_n| = |u_n - u_{N_0+1} + u_{N_0+1}|$
$\leq |u_n - u_{N_0+1}| + |u_{N_0+1}|$
$\leq 1 + |u_{N_0+1}|$.

Taking $M = \max\{|u_0|, |u_1|, |u_2| \dots \dots, |u_{N_0}|, 1 + |u_{N_0+1}|\}$ we get $\forall n \in \mathbb{N}: |u_n| \leq M$. So the sequence (u_n) is bounded.

Hence it has a convergent subsequence, let $(u_{n_k})_{k\in\mathbb{N}}$ be that convergent subsequence, which converges to ℓ. Now, for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that:

$$\forall k \in \mathbb{N}: k > k_0 \Longrightarrow\ |u_{n_k} - \ell| < \frac{\varepsilon}{2}.$$

And the sequence being Cauchy, there exists an $N_1 \in \mathbb{N}$ such that

$$\forall n, p \in \mathbb{N}: n, p > N_1 \Longrightarrow\ |u_n - u_p| < \frac{\varepsilon}{2}.$$

Putting $N = \max\{k_0, N_1\}$ we get $\forall p \in \mathbb{N}$:

$$p > N \Longrightarrow \begin{cases} p > k_0 \Longrightarrow\ |u_{n_p} - \ell| < \dfrac{\varepsilon}{2} \\ p > N_1 \Longrightarrow n_p > N_1 \Longrightarrow\ |u_n - u_{n_p}| < \dfrac{\varepsilon}{2} \quad (\text{Since } n_p \geq p\,). \end{cases}$$

So $\forall \varepsilon > 0; \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}$:

$$\begin{aligned} n > N \Longrightarrow\ |u_n - \ell| &= |u_n - u_{n_p} + u_{n_p} - \ell| \\ &\leq |u_n - u_{n_p}| + |u_{n_p} - \ell| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So the sequence (u_n) is convergent towards ℓ.

Remarks 3.4

1) Sometimes it is easier to prove that a sequence is Cauchy sequence than to prove that it is convergent (i.e. without knowing the limit).

2) A sequence (u_n) is divergent if and only if:

$$\exists\varepsilon > 0; \forall N \in \mathbb{N}; \exists p, q \in \mathbb{N}: (p > N \wedge q > N) \wedge \ |u_p - u_q| \geq \varepsilon.$$

Exercise 3.2

1) Let n_0 be a fixed natural number where $n_0 \geq 1$ and (u_n) a sequence satisfying: $\forall n \geq n_0: |u_{n+1} - u_n| \leq k|u_n - u_{n-1}|$, where k is a fixed real number verified $0 < k < 1$.
Prove that the sequence (u_n) is convergent.

2) Let (a_n) be a sequence defined by $a_0 > 0$; $a_{n+1} = 1 + \frac{1}{a_n}$. Using the first question, study the nature of the sequence (a_n).

Solution

1) We will show that (u_n) is a Cauchy sequence.
First observe that $\forall n \in \mathbb{N}$:

$$\begin{aligned} |u_{n+1} - u_n| &\leq k|u_n - u_{n-1}| \\ &\leq k^2|u_{n-1} - u_{n-2}| \\ &\leq k^3|u_{n-2} - u_{n-3}| \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\leq k^{n-n_0}|u_{n_0+1} - u_{n_0}|. \end{aligned}$$

So $\forall n \geq n_0: |u_{n+1} - u_n| \leq k^{n-n_0}|u_{n_0+1} - u_{n_0}|$.
Next, let $p > n \geq n_0$

$$\begin{aligned} |u_p - u_n| &= |u_p - u_{p-1} + u_{p-1} - u_{p-2} + \ \dots\dots\dots - u_{n+1} + u_{n+1} - u_n| \\ &\leq |u_p - u_{p-1}| + |u_{p-1} - u_{p-2}| + + \ \dots\dots\dots + |u_{n+1} - u_n| \\ &\leq k^{p-1-n_0}|u_{n_0+1} - u_{n_0}| + k^{p-2-n_0}|u_{n_0+1} - u_{n_0}| + \ \dots\dots\dots + k^{n-n_0}|u_{n_0+1} - u_{n_0}| \\ &\leq (k^{n-n_0} + \ \dots\dots\dots + k^{p-2-n_0} + k^{p-1-n_0})|u_{n_0+1} - u_{n_0}|. \\ &\leq k^{n-n_0}\frac{1-k^{p-n}}{1-k}|u_{n_0+1} - u_{n_0}| \\ &\leq \frac{|u_{n_0+1} - u_{n_0}|}{1-k}k^{n-n_0}, \qquad (\text{Since} \quad 1 - k^{p-n} < 1). \end{aligned}$$

We know that $\lim\limits_{n\to\infty} \frac{|u_{n_0+1}-u_{n_0}|}{1-k}k^{n-n_0} = 0$ (Since $0 < k < 1$), therefore,

$\forall\varepsilon > 0; \exists N_0 \in \mathbb{N}; \forall n \in \mathbb{N}: n > N_0 \Longrightarrow \ \frac{|u_{n_0+1}-u_{n_0}|}{1-k}k^{n-n_0} < \varepsilon.$

Putting $N = \max\{N_0, n_0\}$. We get:

$$\forall\varepsilon > 0; \exists N \in \mathbb{N}; \forall p, n \in \mathbb{N}: p > n > N \Longrightarrow |u_p - u_n| \ \leq \frac{|u_{n_0+1} - u_{n_0}|}{1-k}k^{n-n_0} < \varepsilon.$$

So (u_n) is a Cauchy sequence.

2) We first prove by induction that $\forall n \geq 2: a_{n-1} \geq 1$.
Next, we have $\forall n \in \mathbb{N}^*: |a_{n+1} - a_n| = \left|\left(1 + \frac{1}{a_n}\right) - \left(1 + \frac{1}{a_{n-1}}\right)\right|$

$$= \left|\frac{a_n - a_{n-1}}{a_n a_{n-1}}\right|.$$

On the other hand we have,
$\forall n \geq 2: a_n a_{n-1} = 1 + a_{n-1} \geq 1 + 1 = 2$, and $\forall n \geq 2: \frac{1}{a_n a_{n-1}} \leq \frac{1}{2}$.
So

$$\forall n \geq 2: |a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|.$$

Therefore, the sequence (a_n) satisfies the condition given in the first question. Then the sequence (a_n) is convergent.

Exercise 3.3

Let the sequence (u_n) be defined by: $\forall n \in \mathbb{N}^*: u_n = \sum_{k=1}^{n} \frac{1}{k}$.
Prove that the sequence (u_n) is divergent.

Solution

We will show that (u_n) is not a Couchy sequence.
We have $\forall n \in \mathbb{N}^*$:

$$u_{2n+1} - u_{n+1} = \sum_{k=1}^{2n+1} \frac{1}{k} - \sum_{k=1}^{n+1} \frac{1}{k}$$

$$= \sum_{k=n+2}^{2n+1} \frac{1}{k}$$

$$= \sum_{p=2}^{n+1} \frac{1}{n+p}$$

$$\geq \sum_{k=n+2}^{2n+1} \frac{1}{2n+1} \quad \left(\text{Since } \forall n \in \mathbb{N}^*: 2 \leq p \leq n+1 \Rightarrow \frac{1}{n+p} \geq \frac{1}{2n+1}\right)$$

$$\geq n\frac{1}{2n+1} \geq n\frac{1}{2n+n} = \frac{1}{3},$$

so

$$\forall n \in \mathbb{N}^*: u_{2n+1} - u_{n+1} \geq \frac{1}{3}.$$

Putting $q = n+1$, $p = 2n+1$ and $\varepsilon = \frac{1}{3}$, we get,

$$\exists \varepsilon = \frac{1}{3} > 0; \forall n \in \mathbb{N}^*;\ \exists p, q \in \mathbb{N}: (p = 2n+1 > n \wedge q = n+1 > n) \wedge\ \left|u_p - u_q\right| \geq \varepsilon.$$

3.7 Recurrence Sequences

Definition 3.9

Let $f : D \rightarrow \mathbb{R}$ be a function, where $f(D) \subseteq D$ and $\alpha \in D$.We say that the sequence (u_n) is recurrent if it is defined by $u_n = \alpha$ and the recurrent relation:
$\forall n \in \mathbb{N} : u_{n+1} = f(u_n)$.

Monotonicity

The monotonicity of the sequence (u_n) is related to the monotonicity of the function f.
Using proof by induction, the following can be proven true.

Proposition 3.3

1) If f is increasing, the sequence (u_n) is monotonic, increasing if $f(u_0) - u_0 \geq 0$ and decreasing if $f(u_0) - u_0 \leq 0$.
2) If f is decreasing, the sign of the difference $u_{n+1} - u_n$ is alternately negative and positive, which means that (u_n) is non-monotonic in this case.

Proof

1) Assume that f is increasing.
For $f(u_0) - u_0 \geq 0$, let's to prove that $\forall n \in \mathbb{N} : u_{n+1} - u_n \geq 0$. Inded

$$u_1 - u_0 = f(u_0) - u_0 \geq 0.$$

Suppose that $u_{n+1} - u_n \geq 0$, then

$$u_{n+1} - u_n \geq 0 \Longrightarrow u_{n+1} \geq u_n$$

$$\overset{f \text{ is increasing}}{\Longrightarrow} \quad f(u_{n+1}) \geq f(u_n)$$
$$\Longrightarrow u_{n+2} \geq u_{n+1}$$
$$\Longrightarrow u_{n+2} - u_{n+1} \geq 0.$$

In the same way, we prove that: if $f(u_0) - u_0 \leq 0$, then $\forall n \in \mathbb{N} : u_{n+1} - u_n \leq 0$.

2) Assume that f is increasing.

If $u_{n+1} - u_n \geq 0$, then

$$u_{n+1} - u_n \geq 0 \Rightarrow u_{n+1} \geq u_n$$
$$\overset{f \text{ is decreasing}}{\Longrightarrow} \quad f(u_{n+1}) \leq f(u_n)$$
$$\Longrightarrow u_{n+2} \leq u_{n+1}$$
$$\Longrightarrow u_{n+2} - u_{n+1} \leq 0.$$

That is, the sign of the difference $u_{n+1} - u_n$ is alternately negative and positive.

Convergence

Proposition 3.4

We assume that f is continuous on D.

If the sequence (u_n) converges towards ℓ in D, then ℓ is a solution to the equation $f(x) = x$.

Proof

If the sequence (u_n) converges towards ℓ of D then $\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = \ell$.

Since f is continuous at ℓ, then: $\lim_{n\to\infty} f(u_n) = f(\ell)$. On the other hand, we have

$$\forall n \in \mathbb{N} : u_{n+1} = u_n \Rightarrow \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} u_n$$
$$\Rightarrow \lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} u_n$$
$$\Rightarrow f(\ell) = \ell.$$

So, ℓ is a solution of the equation $f(x) = x$.

Remark 3.4

The search for the limit of the sequence (u_n) leads to solving the equation $f(x) = x$, with the unknown x in set D. If the equation has no solution, then the sequence has no limit. However, If the equation has one or more solutions, then the problem returns to studying the possibility that one of these solutions is the limit of the sequence (u_n).

If the equation $f(x) = x$ has solutions, this does not necessarily mean that the sequence (u_n) is convergent.

Example 3.9

Let (u_n) be a sequence defined by $u_0 = a$; $\forall n \in \mathbb{N} : u_{n+1} = \sqrt{2 + u_n}$.

Putting $f(x) = \sqrt{2 + x}$. So the function f is defined continuous and strictly increasing on the domain $D = [-2, +\infty[$, and $f(D) \subseteq D$. Then the sequence (u_n) is defined and monotonic and we have

$$f(u_0) - u_0 = f(a) - a = \sqrt{2 + a} - a = \frac{2 + a - a^2}{\sqrt{2 + a} + a} = \frac{(2 - a)(1 + a)}{\sqrt{2 + a} + a}.$$

So the sign of $f(u_0) - u_0$, is the same sign of $(2 - a)$, also the equation $\sqrt{2 + x} = x$ has a single solution which is 2, hence the following results.

1) If $a < 2$, we can prove that $\forall n \in \mathbb{N} : u_n < 2$. So the sequence (u_n) is increasing and bounded from above by 2.

2) If $a > 2$, we can prove that $\forall n \in \mathbb{N} : u_n > 2$. So the sequence (u_n) is decreasing and bounded from below by 2.

3) If $a = 2$, the sequence (u_n) is constant.

So the sequence (u_n) is convergent in all cases and its limit is 2.

Example 3.10

Let (v_n) be a sequence defined by $v_0 = a > 1; \forall n \in \mathbb{N} : v_{n+1} = {v_n}^2$

Putting $f(x) = x^2$. Since the function f is defined continuous and strictly increasing on the domain $D = [0, +\infty[$, and $f(D) \in D$, and we have $f(v_0) - v_0 = f(a) - a = a^2 - a > 0$

Then the sequence (v_n) is defined and monotonic increasing. And the equation $x^2 = x$ has tow solutions which are $0; 1$, but the sequence (v_n) is divergent. Indeed, by induction we can prove that $\forall n \in \mathbb{N} : v_n = a^{2^n}$, therefore $\lim\limits_{n\to\infty} v_n = +\infty$.

Newton's Method

Newton's method is a technique for generating numerical approximate solutions to equations of the form $f(x) = 0$. If x_n is an approximation of this solution and if $f'(x_n) \neq 0$ the next approximation is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

so (x_n) is a recurrent sequence.

Example 3.11

We can easily get a good approximation to square root of a $(a > 0)$, by applying Newton's method to the equation $f(x) = x^2 - a = 0$.

Since $f'(x) = 2x$, so

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

For $a = 2$, by setting $x_0 = 1$, the following table gives us the first values of x_n and compares them to: $\sqrt{2}$ =1,41421356.....

x_0	x_1	x_2	x_3	x_4
1	$\frac{3}{2}$	$\frac{17}{12}$	$\frac{577}{408}$	$\frac{665857}{470832}$
1.0	1.5	1.41666666..	1.41421568..	1.41421356..

Example 3.12

To approximate the cube root of a $(a > 0)$, applying Newton's method to the equation $f(x) = x^3 - a = 0$. Since $f'(x) = 3x^2$, so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - a}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right).$$

For $a = 2$, by setting $x_0 = 1$, the following table gives us the first values of x_n and compares them to: $\sqrt[3]{2}$ =1,25992104....

x_0	x_1	x_2	x_3	x_4
1	$\frac{4}{3}$	$\frac{91}{72}$	$\frac{1126819}{894348}$	$\frac{2146097524\ 939083451}{1703358734\ 191174242}$
1.0	1.33333333..	1,34722222..	1,25993349...	1,25992105...