Solutions of exercise series number 3

Exercise 01 (The limit, using the definition)

1)

Limit by definition (1)
$\left(\forall \varepsilon \in \mathbb{R}_+^*; \exists \delta \in \mathbb{R}_+^*; \forall x \in V_{x_0}: \lvert x - x_0\rvert < \delta \Longrightarrow \lvert f(x) - \ell\rvert < \varepsilon\right) \Leftrightarrow \left(\lim\limits_{x\to x_0} f(x) = \ell\right).$

Prove that: $\lim\limits_{x\to 3}\frac{x^2-1}{x^2+1} = \frac{4}{5}$.

Let $\varepsilon \in \mathbb{R}_+^*$ and $V_3 =]2,4[$ is an neighbourhood of the number 3. We have:

$$\left|\frac{x^2-1}{x^2+1} - \frac{4}{5}\right| < \varepsilon \Leftrightarrow \frac{1}{5}\left|\frac{x^2-9}{x^2+1}\right| < \varepsilon \Leftrightarrow \frac{1}{5}\frac{\lvert(x+3)\rvert}{x^2+1}\lvert(x-3)\rvert < \varepsilon.$$

On the other hand we have:

$$\forall x \in V_3: \frac{\lvert(x+3)\rvert}{x^2+1} < \frac{7}{5},$$

so

$$\forall x \in V_3: \frac{1}{5}\frac{\lvert(x+3)\rvert}{x^2+1}\lvert(x-3)\rvert < \frac{1}{5}\cdot\frac{7}{5}\lvert(x-3)\rvert = \frac{7}{25}\lvert(x-3)\rvert.$$

Therefore, it is enough to put:

$$\frac{7}{25}\lvert(x-3)\rvert < \varepsilon,$$

so

$$\lvert(x-3)\rvert < \frac{25}{7}\varepsilon.$$

It is enough to choose $\delta = \frac{25}{7}\varepsilon$.

2)

Limit by definition (2)
$(\forall \varepsilon \in \mathbb{R}_+^*; \exists B \in \mathbb{R}_+^*; \forall x \in V_{+\infty}: x > B \Longrightarrow \lvert f(x) - \ell\rvert < \varepsilon) \Leftrightarrow \left(\lim\limits_{x\to +\infty} f(x) = \ell\right).$
$(\forall \varepsilon \in \mathbb{R}_+^*; \exists B \in \mathbb{R}_-^*; \forall x \in V_{-\infty}: x < B \Longrightarrow \lvert f(x) - \ell\rvert < \varepsilon) \Leftrightarrow \left(\lim\limits_{x\to -\infty} f(x) = \ell\right).$

Prove that: $\lim\limits_{x\to +\infty}\frac{x^2-1}{x^2+1} = 1$.

Let $\varepsilon \in \mathbb{R}_+^*$ and $V_{+\infty} =]1, +\infty[$ is an neighbourhood of $+\infty$. We have:

$$\left|\frac{x^2-1}{x^2+1}-1\right|<\varepsilon \Leftrightarrow \left|\frac{2}{x^2+1}\right|<\varepsilon \Leftrightarrow x^2+1>\frac{2}{\varepsilon}$$

On the other hand we have:

$$\forall x \in \mathrm{V}_{+\infty}: x^2+1>x^2$$

Therefore, it is enough to put:

$$x^2>\frac{2}{\varepsilon},$$

so

$$x>\sqrt{\frac{2}{\varepsilon}}$$

It is enough to choose $\mathrm{B}=\sqrt{\frac{2}{\varepsilon}}$.

Prove that: $\lim_{x\to-\infty}\frac{x^2-1}{x^2+1}=1$.

Let $\varepsilon \in \mathbb{R}_+^*$ and $\mathrm{V}_{-\infty}=]-\infty,-1[$ is an neighbourhood of $-\infty$. We have:

$$\left|\frac{x^2-1}{x^2+1}-1\right|<\varepsilon \Leftrightarrow x^2+1>\frac{2}{\varepsilon}.$$

On the other hand we have:

$$\forall x \in \mathrm{V}_{-\infty}: x^2+1>x^2$$

Therefore, it is enough to put:

$$x^2>\frac{2}{\varepsilon},$$

so

$$x<-\sqrt{\frac{2}{\varepsilon}}$$

It is enough to choose $\mathrm{B}=-\sqrt{\frac{2}{\varepsilon}}$.

3)

Limit by definition (3)

$$\left(\forall A \in \mathbb{R}_+^*; \exists \delta \in \mathbb{R}_+^*; \forall x \in \mathrm{V}_{x_0}: |x-x_0|<\delta \Longrightarrow f(x)>A\right) \Leftrightarrow \left(\lim_{x\to x_0} f(x)=+\infty\right).$$

Prove that: $\lim_{x\to 1}\frac{x+2}{(x-1)^2}=+\infty$

Let $A \in \mathbb{R}_+^*$ and $V_3 =]0,2[$ is an neighbourhood of the number 1. We have:

$$\frac{x+2}{(x-1)^2} > A \Leftrightarrow (x-1)^2 \cdot \frac{1}{x+2} < \frac{1}{A}$$

On the other hand we have:

$$\forall x \in V_1 : \frac{1}{x+2} < \frac{1}{2}$$

Therefore, it is enough to put:

$$\frac{1}{2}(x-1)^2 < \frac{1}{A},$$

so

$$|x-1| < \sqrt{\frac{2}{A}}.$$

It is enough to choose $\delta = \sqrt{\frac{2}{A}}$.

4)

Limit by definition (4)

$$(\forall A \in \mathbb{R}_-^* \,;\, \exists B \in \mathbb{R}_+^* ; \forall x \in V_{+\infty} : x > B \Longrightarrow f(x) < A) \Leftrightarrow \left(\lim_{x \to +\infty} f(x) = -\infty\right).$$

Prove that: $\lim_{x \to +\infty} \frac{x^2+x+1}{-x+1} = -\infty$.

Let $A \in \mathbb{R}_-^*$ and $V_{+\infty} =]1, +\infty[$ is an neighbourhood of $+\infty$. We have:

$$\frac{x^2+x+1}{-x+1} = -x - \frac{3}{x-1} - 2.$$

And

$$f(x) < A \Leftrightarrow -x - \frac{3}{x-1} - 2 < A$$

$$\Leftrightarrow x + \frac{3}{x-1} + 2 > -A.$$

On the other hand we have:

$$\forall x \in V_{+\infty} : x + \frac{3}{x-1} + 2 > x + 2.$$

Therefore, it is enough to put:

$$x + 2 > -A,$$

so

$$x > -2 - A$$

It is enough to choose $B = -2 - A$ (where $-2 - A \in \mathbb{R}_+^*$).

<u>**Exercise 02**</u> (Calculation of limits)

1) $\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right) = +\infty - \infty$ I F.

$$\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right) = \lim_{x\to 1}\frac{1}{1-x}\frac{x^2+x-2}{1+x+x^2} = \lim_{x\to 1}\frac{-(x+2)}{1+x+x^2} = -1$$

2) $\lim_{x\to+\infty} x\left(\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right) = +\infty - \infty$ I F. We have:

$$\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x = \frac{\left(\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right)\left(\sqrt{x^2+2x} + 2\sqrt{x^2+x} + x\right)}{\sqrt{x^2+2x} + 2\sqrt{x^2+x} + x}$$

$$= \frac{-2x\left(x - \sqrt{x^2+2x} + 1\right)}{\sqrt{x^2+2x} + 2\sqrt{x^2+x} + x}$$

$$= \frac{-2x\left(x + 1 - \sqrt{x^2+2x}\right)}{\left(\sqrt{x^2+2x} + 2\sqrt{x^2+x} + x\right)}\frac{\left(x + 1 + \sqrt{x^2+2x}\right)}{\left(x + \sqrt{x^2+2x} + 1\right)}$$

$$= \frac{-2x}{\left(\sqrt{x^2+2x} + 2\sqrt{x^2+x} + x\right)\left(x + \sqrt{x^2+2x} + 1\right)}$$

$$= \frac{-2}{\left(\sqrt{1+\frac{2}{x}} + 2\sqrt{1+\frac{1}{x}} + 1\right)\left(1 + \sqrt{1+\frac{2}{x}} + \frac{1}{x}\right)}.$$

So

$$\lim_{x\to+\infty}\left(\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right) = \lim_{x\to+\infty}\frac{-2}{\left(\sqrt{1+\frac{2}{x}} + 2\sqrt{1+\frac{1}{x}} + 1\right)\left(1 + \sqrt{1+\frac{2}{x}} + \frac{1}{x}\right)} = -\frac{1}{4}.$$

3) $\lim_{x\to\mp\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = 1^{\infty}$ I F.

Putting $f(x) = \left(\frac{x^2+1}{x^2-2}\right)^{x^2}$ we get, $\ln\left(f(x)\right) = x^2\ln\left(\frac{x^2+1}{x^2-2}\right)$.

So

$$\lim_{x\to\infty}\ln\left(f(x)\right) = \lim_{x\to\infty} x^2\ln\left(\frac{x^2+1}{x^2-2}\right) = \lim_{x\to\infty}\frac{3x^2}{x^2-2}\frac{\ln\left(1+\frac{3}{x^2-2}\right)}{\frac{3}{x^2-2}}.$$

And by putting $h = \frac{3}{x^2-2}$, then $x \to \infty \iff h \to 0$.

So

$$\lim_{x\to\infty}\ln\left(f(x)\right) = \lim_{x\to\infty}\frac{3x^2}{x^2-2}\cdot\lim_{h\to 0}\frac{\ln(1+h)}{h} = 3.$$

So

$$\lim_{x\to\mp\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \mathrm{e}^3.$$

4) $\lim\limits_{x\to\pm\infty}\frac{x}{\mathrm{E}(x)+1}$. We have:

$$\forall x\in\mathbb{R}:\ \ E(x)\le x<E(x)+1 \Leftrightarrow\ 1+x\ \ge E(x)+1>x$$
$$\Leftrightarrow\ 1+x\ \ge E(x)+1>x.$$

So

For $x>0$, then $1+x\ \ge E(x)+1>x \Leftrightarrow \frac{1}{1+x}\le\frac{1}{E(x)+1}<\frac{1}{x}\Leftrightarrow\frac{x}{1+x}\le\frac{x}{E(x)+1}<1$,

and for $x<-1$, then $1+x\ \ge E(x)+1>x \Leftrightarrow \frac{1}{1+x}\le\frac{1}{E(x)+1}<\frac{1}{x}\Leftrightarrow\frac{x}{1+x}\ge\frac{x}{E(x)+1}>1$.

Since $\lim\limits_{x\to\infty}\frac{x}{x+1}=-$, then $\lim\limits_{x\to\pm\infty}\frac{x}{\mathrm{E}(x)+1}=1$.

Exercise 03

Let the two sequences (x_n) and (x'_n) defined by,

$$\forall n\in\mathbb{N}: x'_n=\frac{1}{2\pi n+\frac{\pi}{2}}\ \text{ and }\ x_n=\frac{1}{2\pi n+\frac{\pi}{6}}.$$

We have

$$\lim_{n\to+\infty}x'_n=\lim_{n\to+\infty}x_n=0,$$

and on the other hand we have:

$$f(x'_n)=\sin\left(2\pi n+\frac{\pi}{2}\right)=1\ \text{ and }\ \ f(x_n)=\sin\left(2\pi n+\frac{\pi}{6}\right)=\frac{1}{2}.$$

So

$$\lim_{n\to+\infty}f(x'_n)=1\ \text{ and }\ \ \lim_{n\to+\infty}f(x_n)=\frac{1}{2},$$

thererfore

$$\lim_{n\to+\infty}f(x'_n)\neq\lim_{n\to+\infty}f(x_n).$$

So the function f does not accept a limit at 0.

4) $f=v\circ u$ and $v(x)=\begin{cases}1\ , & x=0\\ 0\ , & x\neq 0\end{cases}$; $u(x)=x\cos\frac{1}{x}$.

Let the two sequences (x_n) and (x'_n), defined in the previous question. We have

$$\forall n\in\mathbb{N}: u(x_n)=\frac{1}{2\pi n+\frac{\pi}{6}}\cos\left(2\pi n+\frac{\pi}{6}\right)=\frac{\frac{\sqrt{3}}{2}}{2\pi n+\frac{\pi}{6}}\neq 0,$$

and

$$\forall n \in \mathbb{N}: f(x_n) = v\big(u(x_n)\big) = 0,$$

so

$$\lim_{n\to+\infty} f(x_n) = 0.$$

On the other hand we have:

$$\forall n \in \mathbb{N}: u(x_n') = \frac{1}{2\pi n + \frac{\pi}{2}} \cos\left(2\pi n + \frac{\pi}{2}\right) = 0,$$

and

$$\forall n \in \mathbb{N}: f(x_n) = v\big(u(x_n')\big) = 1,$$

so

$$\lim_{n\to+\infty} f(x_n') = 1.$$

thererfore

$$\lim_{n\to+\infty} f(x_n') \neq \lim_{n\to+\infty} f(x_n).$$

So the function f does not accept a limit at 0.

__Exercise 04__

1) $f(x) = \frac{x^3+5x+6}{x^5+1}$; $x_0 = -1$.

$$\begin{aligned}\lim_{x\to-1} f(x) &= \lim_{x\to-1} \frac{x^3+5x+6}{x^5+1} \\ &= \lim_{x\to-1} \frac{(x^2-x+6)(x+1)}{(x^4-x^3+x^2-x+1)(x+1)} \\ &= \lim_{x\to-1} \frac{(x^2-x+6)}{(x^4-x^3+x^2-x+1)} = \frac{8}{5}\end{aligned}$$

So f can be extended by continuing at -1 for the function $\tilde{f}$ where $\tilde{f}(x) = \begin{cases} f(x), & x \neq -1 \\ \frac{8}{5}, & x = -1 \end{cases}$.

3) $f(x) = 1 - x\sin\frac{1}{x}$; $x_0 = 0$. We have

$$\forall x \in \mathbb{R}^*: 0 \leq \left|x \sin\frac{1}{x}\right| < |x|.$$

Since $\lim_{x\to 0}|x| = 0$, then

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

So

$$\lim_{x\to 0} f(x) = 1.$$

So f can be extended by continuing at 0 for the function $\tilde{f}$ where $\tilde{f}(x) = \begin{cases} f(x), & x \neq 0 \\ 1 & , x = 0 \end{cases}$.

Exercise 05

1) $f(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ x^2 + ax + b & \text{if } |x| > 1 \end{cases}$

Since f is continuous at 1 then

$$\lim_{x\to 1^+} f(x) = f(1) \Leftrightarrow \lim_{x\to 1^+} x^2 + ax + b = 1$$

$$\Leftrightarrow a + b = 0.$$

Similarly since f is continuous at 1, then,

$$\lim_{x\to -1^-} f(x) = f(-1) \Leftrightarrow \lim_{x\to -1^-} x^2 + ax + b = -1$$

$$\Leftrightarrow -a + b = -2.$$

So

$$\begin{cases} a + b = 0 \\ -a + b = -2 \end{cases}'$$

therefore

$$\begin{cases} a = 1 \\ b = -1 \end{cases}.$$

Exercise 06 (Application of M.V.T)

Mean Value Theorem
If a function $f\ [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval $]a,b[$, then there exists a point $c \in]a,b[$ such that $f(b) - f(a) = f'(c)(b-a)$.

1) Prove that $\forall x \in]0;1[: 1 + x < e^x < \frac{1}{1-x}$.

By applying the mean value theorem to the function $f(x) = e^x$ on the interval $[a;b] = [0;x]$ where $x \geq 0$, we get:

$$\forall x \geq 0 : \underbrace{e^x}_{f(b)} - \underbrace{e^0}_{f(a)} = f'(c)\left(\underbrace{x}_{b} - \underbrace{0}_{a}\right) \quad , \quad \underbrace{0}_{a} < \text{c} < \underbrace{x}_{b}.$$

So

$$e^x - 1 = e^c \cdot x \quad , \quad 0 < \text{c} < x.$$

We have

$$0 < \text{c} < x \implies e^0 < e^c < e^x \Rightarrow x < e^c x < xe^x \Rightarrow x < e^x - 1 < xe^x$$

So

$$\forall x \geq 0 \ : x < e^x - 1 < xe^x.$$

On the one hand we have:

$$\forall x \geq 0 : x < e^x - 1 \Rightarrow x + 1 < e^x.$$

and on the other hand we have:

$$\forall 0 < x < 1 : e^x - 1 < xe^x \Rightarrow e^x(1 - x) < 1 \Rightarrow e^x < \frac{1}{(1 - x)},$$

So

$$\forall x \in]0;1[\ : 1 + x < e^x < \frac{1}{1 - x}.$$

<u>Exercise 07</u> **(** Hospitat Rule **)**

Hospitat Rule

Let f and g be a continuous functions on a neighbourhood V_a of the point a and differentiable on $V - \{a\}$ then, if the $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then the $\lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$ exists also and

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}.$$

In particular if $f(a) = g(a) = 0$, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

1) $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \frac{0}{0} \ \ I\,F.$

$$\begin{aligned}
\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} &= \lim_{x \to 0} \frac{(e^x - e^{-x} - 2x)'}{(x - \sin x)'} \\
&= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \frac{0}{0} \ \ I\,F \\
&= \lim_{x \to 0} \frac{(e^x + e^{-x} - 2)'}{(1 - \cos x)'} \\
&= \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x} = \frac{0}{0} \ \ I\,F \\
&= \lim_{x \to 0} \frac{(e^x - e^{-x})'}{(\sin x)'} \\
&= \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = 2
\end{aligned}$$

2) $\lim\limits_{x\to\frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x} = \frac{0}{0}\ \ I\ F.$

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x} = \lim_{x\to\frac{\pi}{3}} \frac{\left(\sin\left(x-\frac{\pi}{3}\right)\right)'}{(1-2\cos x)'}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\cos\left(x-\frac{\pi}{3}\right)}{2\sin x}$$

$$= \frac{\cos(0)}{2\sin\frac{\pi}{3}} = \frac{1}{\sqrt{3}}.$$

3) $\lim\limits_{x\to+\infty} x\left(e^{\frac{2x+1}{x^2}}-1\right) = 0.\infty\ I\ F.$

$$\lim_{x\to+\infty} x\left(e^{\frac{2x+1}{x^2}}-1\right) = \lim_{x\to+\infty} \frac{\left(e^{\frac{2x+1}{x^2}}-1\right)}{x}$$

$$= \lim_{x\to+\infty} \frac{\left(e^{\frac{2x+1}{x^2}}-1\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to+\infty} \frac{-\frac{2x+2}{x^3}\, e^{\frac{2x+1}{x^2}}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{2x^3+2x^2}{x^3}\, e^{\frac{2x+1}{x^2}} = 2e^0 = 2$$

6) $\lim\limits_{x\to+\infty} x\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{x}\right)\right] = 0.\infty\ I\ F.$

$$\lim_{x\to+\infty} x\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{x}\right)\right] = \lim_{x\to+\infty} \frac{\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{x}\right)\right]}{\frac{1}{x}}$$

$$= \lim_{x\to+\infty} \frac{\left(\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{x}\right)\right]\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to+\infty} \frac{\pi\left(\tan^2(\frac{\pi}{4}+\frac{\pi}{x})+1\right)}{\tan(\frac{\pi}{4}+\frac{\pi}{x})}$$

$$= \frac{\pi\left(\tan^2\frac{\pi}{4}+1\right)}{\tan\frac{\pi}{4}} = 2\pi.$$

8) $\lim\limits_{x\to0} \frac{3\tan 4x-12\tan x}{3\sin 4x-12\sin x} = \frac{0}{0}\ \ I\ F.$

$$\lim_{x\to 0}\frac{3\tan 4x-12\tan x}{3\sin 4x-12\sin x}=\lim_{x\to 0}\frac{(3\tan 4x-12\tan x)'}{(3\sin 4x-12\sin x)'}$$

$$=\lim_{x\to 0}\frac{3\ \frac{4}{\cos^2 4x}-12\ \frac{1}{\cos^2 x}}{3(4\cos 4x)-12\cos x}$$

$$=\lim_{x\to 0}\frac{\cos^2 x-\cos^2 4x}{\cos^2 x\cos^2 4x(\cos 4x-\cos x)}$$

$$=-\lim_{x\to 0}\frac{(\cos 4x+\cos x)}{\cos^2 x\cos^2 4x}=-2.$$

9) $\lim_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}=\frac{0}{0}$ I F.

$$\lim_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}=\lim_{x\to\frac{\pi}{4}}\frac{\left(2\sqrt{2}-(\cos x+\sin x)^3\right)'}{(1-\sin 2x)'}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{3(\cos x-\sin x)(\cos x+\sin x)^2}{2\cos 2x}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos 2x}\frac{3(\cos x+\sin x)^2}{2}.$$

We have

$$\lim_{x\to\frac{\pi}{4}}\frac{3(\cos x+\sin x)^2}{2}=\frac{3\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)^2}{2}=3,$$

and

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos 2x}=\frac{0}{0}\ I\ F,$$

so

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos 2x}=\lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)'}{(\cos 2x)'}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{-\cos x-\sin x}{-2\sin 2x}=\frac{\sqrt{2}}{2}.$$

So

$$\lim_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}=\frac{\sqrt{2}}{2}3=\frac{3\sqrt{2}}{2}.$$

Exercise 08

f defined on $\mathbb{R}$ by $f(x) = \begin{cases} -\frac{x^2}{x+2} & if \;\; x \geq 0 \\ \ln\frac{2x^2+1}{x^2+1} & if \;\; x < 0 \end{cases}$.

1) Studying the continuity of f over $\mathbb{R}$ leads to studying continuity at 0. So

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{-x^2}{x+2} = 0 = f(0),$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to -} \ln\frac{2x^2+1}{x^2+1} = 0 = f(0).$$

So f is continuous at 0.

2) Studying the continuity of f at 0. We have

$$\lim_{x\to x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to x\to 0^+} \frac{\frac{-x^2}{x+2} - 0}{x-0} = \lim_{x\to x\to 0^+} \frac{-x}{x+2} = 0 = f'_d(0).$$

Similarly we have

$$\lim_{x\to x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to x\to 0^-} \frac{\ln\frac{2x^2+1}{x^2+1}}{x} =$$

$$\lim_{x\to x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to x\to 0^+} \frac{\frac{-x^2}{x+2} - 0}{x-0} = \lim_{x\to x\to 0^+} \frac{-x}{x+2} = 0 = f'_d(0),$$

$$\lim_{x\to x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to x\to 0^-} \frac{\ln\frac{2x^2+1}{x^2+1}}{x} \lim_{x\to 0^-} \frac{\left(\ln\frac{2x^2+1}{x^2+1}\right)'}{(x)'} = \lim_{x\to x\to 0^-} \frac{\frac{2x}{2x^4+3x^2+1}}{1} = 0 = f'_g(0).$$

Since $f'_g(0) = f'_d(0)$ so f is differentiable at 0 and $f'(0) = 0$.

3) Express $f'(x)$ in terms of x.

$$f'(x) = \begin{cases} -\dfrac{x^2+4x}{(x+2)^2} & \text{if } x \geq 0 \\ \dfrac{2x}{2x^4+3x^2+1} & \text{if } x < 0 \end{cases}$$

4) We have $\forall x \in \mathbb{R}^*: f'(x) < 0$, then f is strictly increasing and continuous over $\mathbb{R}$. So f it accepts an inverse function f^{-1} that is continuous and strictly increasing over $f(\mathbb{R}) =]-\infty; \ln 2[$

Express $f^{-1}(x)$ in terms of x. We have $f(\mathbb{R}_-) = [0; \ln 2[$ and $f(\mathbb{R}_+) =]-\infty; 0]$.

For $x \in \mathbb{R}_+$ and $y \in]-\infty; 0]$ then,

$$y = f(x) \Leftrightarrow y = \frac{-x^2}{x+2}$$

$$\Leftrightarrow \begin{cases} x = -\frac{1}{2}y + \frac{1}{2}\sqrt{y(y-8)} \\ x = -\frac{1}{2}y - \frac{1}{2}\sqrt{y(y-8)} \qquad (\text{ unacceptable}). \end{cases}$$

So

$$x = -\frac{1}{2}y + \frac{1}{2}\sqrt{y(y-8)}.$$

For $x \in \mathbb{R}_-$ and $y \in]0;\ln 2[$,

$$y = f(x) \Leftrightarrow y = \ln\frac{2x^2+1}{x^2+1}$$

$$\Leftrightarrow \begin{cases} x = -\sqrt{-\frac{e^y-1}{e^y-2}} \\ x = \sqrt{-\frac{e^y-1}{e^y-2}} \qquad (\text{ unacceptable}). \end{cases}$$

So

$$f^{-1}(x) = \begin{cases} -\frac{1}{2}x + \frac{1}{2}\sqrt{x(x-8)} & \text{if } x \in]-\infty;0] \\ -\sqrt{-\frac{e^x-1}{e^x-2}} & \text{if } x \in]0;\ln 2[. \end{cases}$$

<u>Exercise 11</u>

I)

1) $\tan \operatorname{Arc}\sin x = \frac{x}{\sqrt{1-x^2}}$.

$$\forall x \in]-1,1[: \tan \operatorname{Arc}\sin x = \frac{\sin \operatorname{Arc}\sin x}{\cos \operatorname{Arc}\sin x}$$

$$= \frac{\sin \operatorname{Arc}\sin x}{\sqrt{1-(\sin \operatorname{Arc}\sin x)^2}}.$$

$$= \frac{x}{\sqrt{1-x^2}}.$$

2) $\forall x \geq 0 : Arc\tan(x+1) - Arc\tan x = Arc\tan\frac{1}{1+x+x^2}$.

We have $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \Rightarrow \alpha - \beta = Arc\tan\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

So by putting $\alpha = Arc\tan(x+1)$ and $\beta = Arc\tan x$ we get:

$$\forall x \geq 0 : Arc\tan(x+1) - Arc\tan x = Arc\tan\frac{x+1-x}{1+(x+1)x}$$

$$= \frac{1}{x^2 + x + 1}.$$

3) Since $\sin^2 \alpha = \frac{1}{1+\text{cotan}^2\alpha}$ and for $\alpha \in]0, \pi[$ then, $\sin \alpha = \sqrt{\frac{1}{1+\text{cotan}^2\alpha}}$. So

$$\forall x \in \mathbb{R} : \sin(\text{Arc}\cot\tan x) = \sqrt{\frac{1}{1 + \text{cotan}^2(\text{Arc}\cot\tan x)}} = \frac{1}{\sqrt{1+x^2}}.$$

II)

1) $\lim\limits_{x\to 0} \frac{x \,\text{Arc}\sin x^2}{x\cos x - \sin x} = \frac{0}{0}$ I F.

$$\lim_{x\to 0} \frac{x \,\text{Arc}\sin x^2}{x\cos x - \sin x} = \lim_{x\to 0} \frac{(x \,\text{Arc}\sin x^2)'}{(x\cos x - \sin x)'}$$

$$= \lim_{x\to 0} \frac{\text{Arc}\sin x^2 + \frac{2x^2}{\sqrt{1-x^4}}}{-x\sin x}$$

$$= \lim_{x\to 0} \left(\frac{\text{Arc}\sin x^2}{-x\sin x} + \frac{\frac{2}{\sqrt{1-x^4}}}{-\frac{\sin x}{x}} \right).$$

We have

$$\lim_{x\to 0} \left(\frac{\text{Arc}\sin x^2}{-x\sin x} \right) = \lim_{x\to 0} -\frac{(\text{Arc}\sin x^2)'}{(x\sin x)'}$$

$$= \lim_{x\to 0} -\frac{\frac{2}{\sqrt{1-x^4}}}{\frac{\sin x}{x} + \cos x} = -1,$$

and

$$\lim_{x\to 0} \left(\frac{\frac{2}{\sqrt{1-x^4}}}{-\frac{\sin x}{x}} \right) = -2.$$

So

$$\lim_{x\to 0} \frac{x \,\text{Arc}\sin x^2}{x\cos x - \sin x} = -3.$$

3) $\lim\limits_{x\to 0} (e^x + x)^{\frac{1}{x}} = 1^\infty$ I F.

Putting $f(x) = \ln(e^x + x)^{\frac{1}{x}} = \frac{1}{x}\ln(e^x + x)$. So

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln(e^x + x)}{x} = \frac{0}{0}\ I\ F$$

$$= \lim_{x\to 0} \frac{\left(\ln(e^x + x)\right)'}{(x)'} = \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = 2,$$

so

$$\lim_{x\to 0} (e^x + x)^{\frac{1}{x}} = e^{\left(\lim\limits_{x\to 0} f(x)\right)} = e^2.$$

4) $\lim\limits_{x\to 0} \left(\tan \frac{\pi}{x+4}\right)^{\frac{1}{x}} = 1^\infty$ I F.

Putting $g(x) = \ln\left(\tan \frac{\pi}{x+4}\right)^x = \frac{1}{x} \ln\left(\tan \frac{\pi}{x+4}\right)$. So

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\ln\left(\tan \dfrac{\pi}{x+4}\right)}{x} = \frac{0}{0} \ I\ F.$$

$$= \lim_{x\to 0} \frac{\left[\ln\left(\tan \dfrac{\pi}{x+4}\right)\right]'}{[x]'}$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{\tan \dfrac{\pi}{x+4}} \cdot \dfrac{1}{\cos^2 \dfrac{\pi}{x+4}} \cdot \dfrac{-\pi}{(x+4)^2}}{1}$$

$$= -\pi \cdot \lim_{x\to 0} \frac{1}{\sin \dfrac{\pi}{x+4} \cdot \cos \dfrac{\pi}{x+4} \cdot (x+4)^2}$$

$$= -\pi \frac{1}{\dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{2}}{2} \cdot 4^2} = -\frac{\pi}{8}.$$

So

$$\lim_{x\to 0} g(x) = -\frac{\pi}{8}.$$

And we get

$$\lim_{x\to 0} \left(\tan \frac{\pi}{x+4}\right)^{\frac{1}{x}} = e^{\left(\lim\limits_{x\to 0} g(x)\right)} = e^{-\frac{\pi}{8}} = 1.$$

5) $\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{2\cos x} = \infty^0$ I F.

Putting $h(x) = \ln(\tan x)^{2\cos x} = 2\cos x \ln(\tan x)$. So

$$\lim_{x\to \frac{\pi}{2}^-} h(x) = 2 \lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\tan x)}{\dfrac{1}{\cos x}} = \frac{\infty}{\infty} \ I\ F.$$

$$= 2 \lim_{x\to \frac{\pi}{2}^-} \frac{\left(\ln(\tan x)\right)'}{\left(\dfrac{1}{\cos x}\right)'}$$

$$= 2 \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\cos^2 x \cdot \tan x}}{-\frac{\sin x}{\cos^2 x}}$$

$$= -2 \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\tan x \sin x} = 0.$$

So

$$\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{2\cos x} = e^{\left(\lim_{x \to \frac{\pi}{2}^-} h(x)\right)} = e^0 = 1.$$