TD ~~N°~~ 02

Exo1 study the monotonicity of the following sequences:

① $v_n = \frac{\sqrt{n+1}}{2n} \,/\, n \in \mathbb{N}^*$, ② $w_n = \alpha n + (-1)^n \,/\, n \in \mathbb{N}$, ③ $k_n = (1+\frac{1}{n})^n \,/\, n \in \mathbb{N}^*$

Solution:

① To study the monotonicity of the sequence (v_n), we examine the difference: $v_{n+1} - v_n$.

So, $v_{n+1} - v_n = \frac{\sqrt{n+2}}{2n+2} - \frac{\sqrt{n+1}}{2n} = \frac{1}{2}\left(\frac{\sqrt{n+2}}{n+1} - \frac{\sqrt{n+1}}{n}\right)$

$$= \frac{1}{2\left(\frac{\sqrt{n+2}}{n+1} + \frac{\sqrt{n+1}}{n}\right)} \cdot \left(\frac{n+2}{(n+1)^2} - \frac{(n+1)}{n^2}\right) = \frac{1}{2\left(\frac{\sqrt{n+2}}{n+1} + \frac{\sqrt{n+1}}{n}\right)} \cdot \left(\frac{-n^2-3n-1}{n^2(n+1)^2}\right)$$

$$= - \underbrace{\frac{1}{2\left(\frac{\sqrt{n+2}}{n+1} + \frac{\sqrt{n+1}}{n}\right)} \cdot \left(\frac{n^2+3n+1}{n^2(n+1)^2}\right)}_{>0} < 0$$

$\Rightarrow v_{n+1} < v_n$ for any $n \in \mathbb{N}^*$ $\Rightarrow (v_n) \downarrow$ (strictly monotonic decreasing)

② For the sequence (w_n), we examine the sign of the difference $w_{n+1} - w_n$.

So, $w_{n+1} - w_n = \alpha(n+1) + (-1)^{n+1} - (\alpha n + (-1)^n) = \alpha n + \alpha - (-1)^n - \alpha n - (-1)^n$

$$= \alpha - 2(-1)^{\alpha} = \begin{cases} \alpha - 2 & \text{if } n \text{ is even } (n = 2k); \\ \alpha + 2 & \text{if } n \text{ is odd } (n = 2k+1). \end{cases}$$

Now, If $\alpha \leq -2$, then $\alpha - 2 \leq 0$ and $\alpha + 2 \leq 0$

thus $w_{n+1} - w_n \leq 0$; for any $n \in \mathbb{N}$

$\Rightarrow (w_n) \downarrow$ (monotonic decreasing).

α		-2		$+2$	
$\alpha - 2$	$-$		$-$	0	$+$
$\alpha + 2$	$-$	0	$+$		$+$

If $\alpha \geq 2$, then we have $\alpha - 2 \geq 0$, $\alpha + 2 > 0$

thus $w_{n+1} - w_n \geq 0$; for any $n \in \mathbb{N}$

$\Rightarrow (w_n) \nearrow$ (monotonially increasing).

If $-2 < \alpha < 2$, in this case, the sign of $(w_{n+1} - w_n)$ oscillates $\Rightarrow (w_n)$ is not monotone.

3) For the sequence (k_n), we examine the sign of the division $\frac{k_{n+1}}{k_n}$.

So, $$\frac{k_{n+1}}{k_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+1}{n}\right)\frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^{n+1}}$$

$$= \left(\frac{n+1}{n}\right)\left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} = \left(\frac{n+1}{n}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n^2+2n+1-1}{n^2+2n+1}\right)^{n+1}$$

$$= \left(\frac{n+1}{n}\right)\left(1+\frac{-1}{n^2+2n+1}\right)^{n+1}$$

By using Bernoulli's inequality (i.e., $\forall a \geq -1,\ \forall m \in \mathbb{N}: (1+a)^m \geq 1+ma$)

we get: $$\frac{k_{n+1}}{k_n} = \left(\frac{n+1}{n}\right)\left(1+\frac{-1}{n^2+2n+1}\right)^{n+1} \geq \left(\frac{n+1}{n}\right)\left(1+(n+1)\cdot\frac{(-1)}{n^2+2n+1}\right)$$

(Notice that $\frac{-1}{n^2+2n+1} \geq -1$)

therefore $$\frac{k_{n+1}}{k_n} \geq \left(\frac{n+1}{n}\right)\left(1-\frac{(n+1)}{n^2+2n+1}\right) = \left(\frac{n+1}{n}\right)\left(1-\frac{1}{n+1}\right) = \frac{n+1}{n}-\frac{1}{n} = 1$$

$\Rightarrow \frac{k_{n+1}}{k_n} \geq 1$, for any $n \in \mathbb{N}^*$ $\Rightarrow k_{n+1} \geq k_n$, for any $n \in \mathbb{N}^*$

$\Rightarrow (k_n) \nearrow$ (~~strictly~~ monotonic increasing)

Exo2 Calculate the limit of each of the following sequences:

① $a_n = \frac{3^n+(-1)^n}{2^n+2(-1)^n}$ / $n \in \mathbb{N}$, ② $b_n = \frac{1+2+\dots+(2n-1)}{n^2+n}$ / $n \in \mathbb{N}^*$

③ $d_n = \left(1+\frac{1}{n}\right)^n$ / $n \in \mathbb{N}^*$, ④ $g_n = \frac{1+2+2^2+\dots+2^n}{2^n}$ / $n \in \mathbb{N}$.

Solution:

① We have $a_n = \frac{3^n+(-1)^n}{2^n+2(-1)^n} = \frac{\left(\frac{3}{2}\right)^n+\left(\frac{-1}{2}\right)^n}{1+2\left(\frac{-1}{2}\right)^n} \Rightarrow \lim a_n = +\infty$

(Notice that: if $|a|<1 \Rightarrow \lim a^n = 0$;
if $a>1 \Rightarrow \lim a^n = +\infty$)

② We have $b_n = \dfrac{1+3+\cdots+(2n-1)}{n^2+n} = \dfrac{\frac{n}{2}(1+2n-1)}{n^2+n} = \dfrac{n^2}{n^2+n}$

$\Rightarrow \lim\limits_{n\to+\infty} b_n = 1.$

③ We have $\lim\limits_{n\to+\infty} \ln(d_n) = \lim\limits_{n\to+\infty}\left(n\ln\left(1+\frac{1}{n}\right)\right) = \lim\limits_{n\to+\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$

Here, we have applied the well-known limit $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1.$

Now, the limit $\lim\limits_{n\to+\infty} \ln(d_n) = 1$, gives us $\ln\left(\lim\limits_{n\to+\infty} d_n\right) = 1$

(because the function $\ln$ is continuous near e)

which, in turn yields $\lim\limits_{n\to+\infty} d_n = e.$

④ We have $g_n = \dfrac{1+2+2^2+\cdots+2^n}{2^n} = \dfrac{\frac{2^{n+1}-1}{2-1}}{2^n} = \dfrac{2^{n+1}-1}{2^n} = 2 - \dfrac{1}{2^n}$

$\Rightarrow \lim g_n = 2.$

Exo3: ① Using the squeeze theorem, calculate the limit of each of the following sequences: ⓑ $v_n = \dfrac{1}{n^2}\sum\limits_{k=1}^{n} E(kx)$ / $n \in \mathbb{N}^*$, $x \in \mathbb{R}$

ⓒ $w_n = \dfrac{n!}{n^n}$ / $n \in \mathbb{N}^*$.

② Prove using (ε, δ)-definition that: ⓐ $\lim(\sqrt{n+1} - \sqrt{n}) = 0$,

ⓑ $\lim \dfrac{n^2-1}{2n^2+n} = 2$, ⓓ $\lim a^n = +\infty$, where $a > 1$, $n \in \mathbb{N}$.

ⓕ $\lim \dfrac{2^n + (-1)^n}{2^n} = 1$ / $n \in \mathbb{N}$.

Solution: Part ①

ⓐ For any $n \in \mathbb{N}^*$ and $k \in \{1, 2, \ldots, n\}$, we have: $E(kx) \leq kx < E(kx)+1$,

$$\left.\begin{array}{l} \text{RHS} \Rightarrow kx - 1 < E(kx) \\ \text{LHS} \Rightarrow E(kx) \leq kx \end{array}\right\} \Rightarrow kx - 1 < E(kx) \leq kx \text{, for any } k \in \{1, 2, \ldots, n\}$$

we sum the n last inequalities (for $k = 1, 2, \ldots, n$), we get:

$$\sum_{k=1}^{n}(kx-1) < \sum_{k=1}^{n} E(kx) \leq \sum_{k=1}^{n} kx$$

which yields: $\frac{n}{2}(x-1+nx-1) < \sum_{k=1}^{n} E(kx) \leq \frac{n}{2}(x+nx)$

therefore: $\left(\frac{n(n+1)}{2}x - n\right) < \sum_{k=1}^{n} E(kx) \leq \frac{n(n+1)}{2}x$

thus: $\frac{(n+1)}{2n}x - \frac{1}{n} < \frac{1}{n^2}\sum_{k=1}^{n} E(kx) \leq \frac{(n+1)}{2n}x$

since $\lim\left(\frac{(n+1)}{2n}x - \frac{1}{n}\right) = \frac{1}{2}x$ and $\lim\left(\frac{(n+1)}{2n}x\right) = \frac{1}{2}x$, we conclude that:

$\lim\left(\frac{1}{n^2}\sum_{k=1}^{n} E(kx)\right) = \frac{1}{2}x.$

(c) For any $n \in \mathbb{N}^*$ and $k \in \{2, 3, \dots, n\}$, we have $n \geq k$,

therefore: $\prod_{k=2}^{n} n \geq \prod_{k=2}^{n} k$, ($\prod_{k=2}^{n}$ symbolizes multiplication)

which leads to: $n^{n-1} \geq n!$,

thus: $\frac{n!}{n^n} \leq \frac{1}{n}$ that is $0 < w_n \leq \frac{1}{n}$; for any $n \in \mathbb{N}^*$.

Now, since $\lim \frac{1}{n} = 0$, we obtain that: $\lim w_n = 0$.

Part2

Be sure to keep the following equivalence definition in mind:

$\lim U_n = +\infty$ ((U_n) divergent) $\iff \forall A>0, \exists N \in \mathbb{N};\ \forall n \in \mathbb{N}: (n>N \Rightarrow U_n > A)$,

$\lim U_n = -\infty$ ((U_n) divergent) $\iff \forall B<0, \exists N \in \mathbb{N},\ \forall n \in \mathbb{N}: (n>N \Rightarrow U_n < B)$,

$\lim U_n = l$ (for "l" finite) ((U_n) convergent to l) $\iff \forall \varepsilon>0, \exists N \in \mathbb{N},\ \forall n \in \mathbb{N}: (n>N \Rightarrow |U_n - l| < \varepsilon)$.

(a) Let $\varepsilon > 0$, we search for a suitable N such that:

$$\forall n \in \mathbb{N}: (n > N \Rightarrow |\sqrt{n+1} - \sqrt{n}| < \varepsilon).$$

First, notice that: $|\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n+1} - \sqrt{n}} < \frac{1}{2\sqrt{n}}$,

also, $\frac{1}{2\sqrt{n}} < \varepsilon$ iff $n > \frac{1}{4\varepsilon^2}$.

($[\frac{1}{4\varepsilon^2}]$ symbolizes the integer part of $\frac{1}{4\varepsilon^2}$)

So, it suffice to choose $N \geq [\frac{1}{4\varepsilon^2}] + 1$ to obtain:

$$\begin{aligned}\forall n \in \mathbb{N}:\ n > N &\Rightarrow n > \frac{1}{4\varepsilon^2}\\ &\Rightarrow \frac{1}{2\sqrt{n}} < \varepsilon\\ &\Rightarrow |\sqrt{n+1} - \sqrt{n}| < \varepsilon\end{aligned}$$, as required

(b) Let any $\varepsilon > 0$, we're looking for a suitable N s.t.:

$$\forall n \in \mathbb{N} : \left(n > N \Rightarrow \left| \frac{n^2-1}{2n^2+n} - \frac{1}{2} \right| < \varepsilon \right)$$

We know that: $\left| \frac{n^2-1}{2n^2+n} - \frac{1}{2} \right| = \left| \frac{2(n^2-1) - 2n^2 - n}{2(2n^2+n)} \right| = \left| \frac{-2-n}{2(2n^2+n)} \right|$

$$= \frac{2+n}{4n^2+2n} < \frac{4n+2}{4n^2+2n} = \frac{2n+1}{2n^2+n} = \frac{1}{n}$$

also, $\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$.

So, it is enough to choose $N \geq [\frac{1}{\varepsilon}] + 1$ s.t.

$$\forall n \in \mathbb{N} : \quad n > N \Rightarrow n > \frac{1}{\varepsilon}$$
$$\Rightarrow \frac{1}{n} < \varepsilon$$
$$\Rightarrow \left| \frac{n^2-1}{2n^2+n} - \frac{1}{2} \right| < \varepsilon$$

as was to be shown.

(d) Let any $A > 0$, the goal is to discover a suitable N, s.t:

$$\forall n \in \mathbb{N} : n > N \Rightarrow a^n > A$$

Since $a > 1$, we put: $a = 1 + \alpha$, for some $\alpha > 0$.

According to the Bernoulli's inequality, we obtain:

$$a^n = (1+\alpha)^n \geq 1 + n\alpha.$$

Also, it is evident that:

$$1 + n\alpha > A \iff n\alpha > A - 1$$
$$\iff n > \frac{A-1}{\alpha}.$$

So, it suffice to choose $N \geq [\frac{A-1}{\alpha}] + 1$, to obtain:

$$\forall n \in \mathbb{N} : n > N \Rightarrow n > \frac{A-1}{\alpha}$$
$$\Rightarrow 1 + n\alpha > A$$
$$\Rightarrow a^n > A \quad , \quad \text{as desired.}$$

(f) Let any $\varepsilon>0$, we aim to identify an N that satisfies:

$$\forall n \in \mathbb{N}: \left(n>N \Rightarrow \left|\frac{2^n+(-1)^n}{2^n}-1\right|<\varepsilon\right)$$

We have: $$\left|\frac{2^n+(-1)^n}{2^n}-1\right| = \frac{|2^n+(-1)^n-2^n|}{2^n} = \frac{1}{2^n}$$

According to the Bernoulli's inequality, we obtain:

$$2^n=(1+1)^n \geq 1+n, \text{ so } \frac{1}{2^n} \leq \frac{1}{1+n},$$

thus: $$\left|\frac{2^n+(-1)^n}{2^n}-1\right| = \frac{1}{2^n} < \frac{1}{1+n}; \quad \text{for any } n\in\mathbb{N}^*$$

It's evident that: $$\frac{1}{n+1}<\varepsilon \Leftrightarrow n>\frac{1}{\varepsilon}-1.$$

therefore, it suffice to choose $N \geq \left[\frac{1}{\varepsilon}-1\right]+1$, to get:

$$\begin{aligned} \forall n\in\mathbb{N}: \quad n>N &\Rightarrow n>\frac{1}{\varepsilon}-1 \\ &\Rightarrow \frac{1}{n+1}<\varepsilon \\ &\Rightarrow \left|\frac{2^n+(-1)^n}{2^n}-1\right|<\varepsilon, \end{aligned}$$

as required.

Part (3): Prove that the sequence (u_n) is divergent in each of the following cases: (a) $u_n=(-1)^n\frac{n+2}{n}$, $n\in\mathbb{N}^*$;

Solution:

(a) To establish the divergence of a sequence (u_n), it suffices to either extract a divergent subsequence from it, or extract two distinct subsequences from it, that converge to different limits.

To do this, let two subsequences $u_{2n}=\frac{2n+2}{2n}$ and $u_{2n+1}=-\frac{2n+3}{2n+1}$ for $n\in\mathbb{N}^*$

we have: $\lim u_{2n}=1$ and $\lim u_{2n+1}=-1$, therefore (u_n) is divergent.