TD №01

Exercise №01

① Prove that: $\forall x, y \in \mathbb{R} \quad \big||x| - |y|\big| \leq |x+y|$?

Let $x, y \in \mathbb{R}$, we have

$$|x| = |x+y-y| \leq |x+y| + |-y| \Rightarrow |x| - |y| \leq |x+y|$$

$$|y| = |y+x-x| \leq |x+y| + |-x| \Rightarrow |y| - |x| \leq |x+y|$$

thus $\quad -|x+y| \leq |x| - |y| \leq |x+y|$

So, $\quad \big||x| - |y|\big| \leq |x+y| \quad$ for all $x, y \in \mathbb{R}$.

② Prove that: $\forall x, y \in \mathbb{R}, \forall \varepsilon > 0 : xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2} y^2$?

Let $x, y \in \mathbb{R}$ and $\varepsilon > 0$

We have $\quad (a-b)^2 \geq 0 \quad$ for all $a, b \in \mathbb{R}$

So, $\quad a^2 + b^2 \geq 2ab \quad$ for all $a, b \in \mathbb{R}$.

by Putting $a = \frac{x}{\sqrt{\varepsilon 2}}$, $b = \sqrt{\frac{\varepsilon}{2}}$, we get:

$$xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2} y^2.$$

③ Prove that: $(\forall \varepsilon > 0 : |x| < \varepsilon) \Rightarrow (x = 0)$?

We use the contrapositive proof, which means that; it is sufficient to prove that:

$$(x \neq 0) \Rightarrow (\exists \varepsilon > 0 : |x| \geq \varepsilon) \ ?$$

so, to prove the statement $(\exists \varepsilon > 0 : |x| \geq \varepsilon)$ is true for $x \neq 0$, it suffices to choose $\varepsilon = \frac{|x|}{2}$.

④ Prove that: $\forall x, y \in \mathbb{R}: \ |x|+|y| \leq |x+y|+|x-y|$?

We have $|a+b| \leq |a|+|b|$, $|a-b| \leq |a|+|b|$, for all $a, b \in \mathbb{R}$.

then by summation, we obtain the following:

$$|a+b|+|a-b| \leq 2|a|+2|b| \quad \cdots \text{①}$$

Now, by substituting $a=\frac{x+y}{2}$, $b=\frac{x-y}{2}$ into the last inequality ①,

we get: $\quad |x|+|y| \leq |x+y|+|x-y|$ as the desired result.

⑤ Prove that: $\left(|x+y|=|x|+|y|\right) \Leftrightarrow (xy \geq 0)$

Let x, y be two real numbers, we have:

$$\begin{aligned} |x+y|=|x|+|y| \quad &\text{iff} \quad |x+y|^2 = (|x|+|y|)^2 \\ &\text{iff} \quad x^2+y^2+2xy = x^2+y^2+2|xy| \\ &\text{iff} \quad xy=|xy| \\ &\text{iff} \quad xy \geq 0. \end{aligned}$$

Hence, the desired results.

Exercise 2

Part ①

ⓐ Prove that: $\quad \forall \varepsilon>0, \exists n \in \mathbb{N}^*: 0<\frac{1}{n}<\varepsilon$

By setting $x=\varepsilon$, $y=1$ into the Archimedes' axiom:

$$\forall x>0, \forall y \in \mathbb{R}, \exists n \in \mathbb{N}^*: nx>y$$

We obtain:

$$\forall \varepsilon>0, \exists n \in \mathbb{N}^*: n\varepsilon>1$$

which yields:

$$\forall \varepsilon>0, \exists n \in \mathbb{N}^*: 0<\frac{1}{n}<\varepsilon$$

Hence, the required inequality.

(b) Prove that: $\forall x, y \in \mathbb{R}: (x<y) \Rightarrow (E(x) \leq E(y))$.

Let x, y be two real numbers.

If we have $x<y$, then $E(x) \leq x < y$, and since $E(y)$ is the largest integer less than or equal to y, we get: $E(x) \leq E(y)$.

(c)* Prove that: $\forall x, y \in \mathbb{R}: E(x)+E(y) \leq E(x+y) \leq E(x)+E(y)+1$.

(d) Prove that: $\forall x \in \mathbb{R}: -1 \leq E(x)+E(-x) \leq 0$.

Let x is a real number, we substitute $y=-x$ into the following inequality:

$$\forall x, y \in \mathbb{R}: E(x)+E(y) \leq E(x+y) \leq E(x)+E(y)+1$$

we get:

$$\forall x \in \mathbb{R}: \quad E(x)+E(-x) \leq E(0) \leq E(x)+E(-x)+1.$$

therefore:
$$\begin{cases} E(x)+E(-x) \leq 0; \\ E(x)+E(-x) \geq -1. \end{cases}$$

(e) Prove that: $\forall x, y \in \mathbb{R};\ Max\{x, y\} = \dfrac{x+y+|x-y|}{2},\ Min\{x, y\} = \dfrac{x+y-|x-y|}{2}$.

Let $x, y \in \mathbb{R}$, we have:

$$x+y = Max\{x, y\} + Min\{x, y\} \quad \cdots (1)$$

$$|x-y| = Max\{x, y\} - Min\{x, y\} \quad \cdots (2)$$

by summing and subtracting the last two equalities (1) and (2), we get:

$$Max\{x, y\} = \frac{x+y+|x-y|}{2} \quad \text{and}$$

$$Min\{x, y\} = \frac{x+y-|x-y|}{2} \quad \text{respectively.}$$

Exercise 2 :

Part 2 : Determine, if possible the Maximum, minimum, supremum and Infimum for each set from the following :

ⓐ $A = \{ \frac{2n+1}{n} / n \in \mathbb{N}^* \}$ ⓑ $B = \{ \frac{1}{1+x^2} / x \in \mathbb{R} \}$.

Solution:

ⓐ first of all the set A is not empty (because $3 = \frac{2(1)+1}{1} \in A$).

We must now show whether the set A is bounded from below and above or not :

We have : $\forall n \in \mathbb{N}^* : \frac{2n+1}{n} = 2 + \frac{1}{n}$

then : $2 \leq 2 + \frac{1}{n} \leq 3$ ... ①

We can only say that 3 is an upper bound for the set A, but in this case, since $3 \in A$, then $Max(A) = 3$, and therefore $Sup(A) = 3$, and we don't need to use the characteristic property of sup.

Now, from the inequality ①, we deduce that A is bounded from below by 2.

to prove that $\inf(A) = 2$ by Archimedes' axiom, we use :

$$\forall \varepsilon > 0, \exists n \in \mathbb{N}^* : \varepsilon n > 1$$

So : $\forall \varepsilon > 0, \exists n \in \mathbb{N}^* : 0 < \frac{1}{n} < \varepsilon$

then : $\forall \varepsilon > 0, \exists n \in \mathbb{N}^* \quad \frac{1}{n} + 2 < 2 + \varepsilon$

thus : $\forall \varepsilon > 0, \exists a = \frac{1}{n} + 2 \in A : 2 < a < 2 + \varepsilon$.

therefore $Inf(A) = 2$.

Since $2 \notin A$, we conclude that the $Min(A)$ does not exist.

(5) Notice that $B=\{\frac{1}{x^2+1} \mid x\in\mathbb{R}\}$ is not empty (because $1=\frac{1}{0+1}\in B$)

Now, we have: $\forall x\in\mathbb{R}: x^2\geq 0$, which yields that: $0<\frac{1}{x^2+1}\leq 1$

So, the set B is Bounded from above by 1 and since $1\in B$, we conclude

that: $Max(B)=1=Sup(B)$.

We have 0 is a lower bound of B and $0\notin B$, then $Min(B)\neq 0$.

to prove that: $\inf(B)=0$, we use the characteristic property of Inf.

that is: $\forall\varepsilon>0, \exists b\in B: 0<b<0+\varepsilon$, or equivalently, that:

$$\forall\varepsilon>0, \exists x\in\mathbb{R}: \quad 0<\frac{1}{x^2+1}<\varepsilon.$$

On the other hand:

$$\frac{1}{x^2+1}<\varepsilon \iff x^2>\frac{1}{\varepsilon}-1.$$

thus, if $\varepsilon\geq 1$ (i.e., $\frac{1}{\varepsilon}-1\leq 0$), then we can choose any $x\in\mathbb{R}^*$.

If $\varepsilon<1$ (i.e., $\frac{1}{\varepsilon}-1>0$), then $(x^2>\frac{1}{\varepsilon}-1) \iff (x>\sqrt{\frac{1}{\varepsilon}-1}$ or $x<-\sqrt{\frac{1}{\varepsilon}-1})$

in this case, it is sufficient to choose x from the interval $]-\infty, -\sqrt{\frac{1}{\varepsilon}-1}[$

or $]\sqrt{\frac{1}{\varepsilon}-1}, +\infty[$.

Exercise 3

Part (1)

(a) Prove that if the natural number n is not a perfect square, then $\sqrt{n}$ is irrational.

Solution: Assume that n is a natural number that is not a perfect square i.e., $\forall m\in\mathbb{N}: n\neq m^2$. We will prove that $\sqrt{n}\notin\mathbb{Q}$

- Reasoning by absurd (contraduction). Suppose this fails. that is $\sqrt{n}=\frac{p}{q}$ for some coprime integers p and q ($\gcd(p,q)=1$)

thus: $\sqrt{n} = \frac{p}{q} \Leftrightarrow p^2 = nq^2$

$\Leftrightarrow q^2 \backslash p^2$ (q^2 divise p^2)

$\Rightarrow q^2 = 1$ (gcd $(p,q) = 1$)

$\Rightarrow p^2 = n$ and this contradict the assumption

n is not a perfect square, hence $\sqrt{n} \notin \mathbb{Q}$.

ⓑ Prove that: if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$

Solution:

We use contradiction; suppose that $r + x \in \mathbb{Q}$.

from $r \in \mathbb{Q}$, we have $-r \in \mathbb{Q}$, thus $(r+x) + (-r) \in \mathbb{Q}$; that is $x \in \mathbb{Q}$

and this contradict the assumption $x \notin \mathbb{Q}$, therefore $r + x \notin \mathbb{Q}$

ⓒ Prove that: If $r \in \mathbb{Q}^*$ and $x \notin \mathbb{Q}$, then $rx \notin \mathbb{Q}$.

Solution: Applying the same methods.

ⓓ Explain why the number $\sqrt{15} + \sqrt{12} \notin \mathbb{Q}$?

Solution:

Reasoning by absurd; assume that $\sqrt{15} + \sqrt{12} \in \mathbb{Q}$, then $(\sqrt{15} + \sqrt{12})^2 \in \mathbb{Q}$

that is $27 + 2\sqrt{180} \in \mathbb{Q}$ so $\sqrt{180} \in \mathbb{Q}$. But 180 is not a perfect square, which means that $\sqrt{180} \notin \mathbb{Q}$, this gives contradiction.

Therefore $\sqrt{15} + \sqrt{12} \notin \mathbb{Q}$.

Exercise 4:

Part ① Let E and F be two non-empty and bounded sets.

ⓐ Prove that: $(E \subseteq F) \Rightarrow (Inf(F) \leq Inf(E) \leq Sup(E) \leq Sup(F))$

Solution:

Assume that $E \subseteq F$, from the fact $\forall x \in F : x \geq Inf(F)$, we get:

$\forall x \in E : x \geq Inf(F)$, that is $Inf(F)$ is a lower bound for E

and since $Inf(E)$ is the greatest lower bound for E, we obtain:

$Inf(F) \leq Inf(E)$.

ⓑ Prove that: $Sup(E \cup F) = Max\{Sup(E), Sup(F)\}$.

Solution:

We have: $x \in E \cup F \iff x \in E$ or $x \in F$

then $x \leq Sup(E)$ or $x \leq Sup(F)$

therefore $\forall x \in E \cup F : \quad x \leq Max\{Sup(E), Sup(F)\}$

so, $\quad Sup(E \cup F) \leq Max\{Sup(E), Sup(F)\} \quad \cdots ①$

on the other hand:

$E \subseteq E \cup F \Rightarrow Sup(E) \leq Sup(E \cup F)$

$F \subseteq E \cup F \Rightarrow Sup(F) \leq Sup(E \cup F)$

thus $Max\{Sup(E), Sup(F)\} \leq Sup(E \cup F) \quad \cdots ①$

from the inequalities ① and ②, we get:

$$Sup(E \cup F) = Max\{Sup(E), Sup(F)\}.$$

Part ② We set: $E - F = \{x - y \,/\, x \in E \text{ and } y \in F\}$, $-F = \{-x \,/\, x \in F\}$

ⓐ Prove that $Sup(E - F) = Sup(E) - Inf(F)$.

Solution:

We have $Sup(E) = M \iff \begin{cases} \forall x \in E : \quad x \leq M \text{ and} & \cdots ① \\ \forall \varepsilon > 0, \exists a \in E : \quad a \geq M - \frac{\varepsilon}{2} & \cdots ② \end{cases}$

also,

$$\text{Inf}(F)=m \iff \begin{cases} \forall y\in F: \quad y\geq m & \cdots (3)\\ \forall \varepsilon>0, \exists b\in F: \quad b\leq m+\frac{\varepsilon}{2} & \cdots (4)\end{cases}$$

$$\iff \begin{cases} \forall y\in F: \quad -y\leq -m & \cdots (5)\\ \forall \varepsilon>0, \exists b\in F \quad -b\geq -m-\frac{\varepsilon}{2} & \cdots (6)\end{cases}$$

By adding the inequalities (1) and (5) as well as the inequalities (2) and (6), we get:

$$\begin{cases} \forall x\in E, \forall y\in F: \quad x-y\leq M-m\\ \forall \varepsilon>0, \exists a\in E, \exists b\in F: \quad a-b\geq M-m-\varepsilon\end{cases}$$

thus $\text{Sup}(E-F)=M-m=\text{Sup}(E)-\text{Inf}(F)$.

(b) Similarly, we can prove that: $\text{Inf}(E-F)=\text{Inf}(E)-\text{Sup}(F)$,

(c) Prove that: $\text{Sup}(-F)=-\text{Inf}(F)$.

Solution:

Put $E=\{0\}$ in the relation: $\text{Sup}(E-F)=\text{Sup}(E)-\text{Inf}(F)$, we get:

$\text{Sup}(-F)=-\text{Inf}(F)$. (because $\{0\}-F=-F$)

Part (3) Let $E\subseteq \mathbb{R}_+^*$ and $1/E=\{\frac{1}{x} \mid x\in E\}$

(a) Prove that: $\text{Inf}(1/E)=\dfrac{1}{\text{Sup}(E)}$

Solution:

We have: $\forall x\in E: x\leq \text{Sup}(E)$, then $\frac{1}{x}\geq\frac{1}{\text{Sup}(E)}$, it follows:

$\forall(\frac{1}{x})\in 1/E: \frac{1}{x}\geq\frac{1}{\text{Sup}(E)}$, and since $\text{Inf}(1/E)$ is the greatest lower bound for $1/E$, we obtain $\text{Inf}(1/E)\geq\dfrac{1}{\text{Sup}(E)}$ $\cdots$ (1)

On the other hand, we have: $\forall(\frac{1}{x})\in 1/E: \frac{1}{x}\geq \text{Inf}(1/E)$ which means $\forall x\in E: x\leq\dfrac{1}{\text{Inf}(1/E)}$ and since $\text{Sup}(E)$ is the least upper bound for E we get $\text{Sup}(E)\leq\dfrac{1}{\text{Inf}(1/E)}$ $\cdots$ (2). From (1) and (2) we get: $\text{Sup}(E)=\dfrac{1}{\text{Inf}(1/E)}$.