TD (complex Variable)

Exercise 1:

- Write $\cos^5(x)$ in linear form?

Solution:

According the formula : $\forall x \in \mathbb{R}, \forall k \in \mathbb{Z} : \cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$

for $k=1$, we have $\cos^5(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^5 = \sum_{m=0}^{5} C_5^m e^{i(5-m)x} e^{-imx}$

Using the Pascale's triangle to calculate C_5^m.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
(n=5) 1 5 10 10 5 1

$\Rightarrow \left\{ C_5^0 = 1, C_5^1 = 5, C_5^2 = 10, C_5^3 = 10, C_5^4 = 5, C_5^5 = 1 \right\}$

Thus $\cos^5(x) = \frac{1}{2^5}\left(e^{i5x} + 5e^{i4x}e^{-ix} + 10e^{i3x}e^{-i2x} + 10e^{i2x}e^{-i3x} + 5e^{ix}e^{-i4x} + e^{-i5x}\right).$

$$= \frac{1}{2^5}\left(e^{i5x} + 5e^{i3x} + 10e^{ix} + 10e^{-ix} + 5e^{-i3x} + e^{-i5x}\right)$$

$$= \frac{1}{2^5}\left(\left(e^{i5x} + e^{-i5x}\right) + 5\left(e^{i3x} + e^{-i3x}\right) + 10\left(e^{ix} + e^{-ix}\right)\right)$$

$$= \frac{1}{2^5}\left(2\cos(5x) + (5)(2)\cos(3x) + (10)(2)\cos(x)\right)$$

$$= \frac{1}{16}\cos(5x) + \frac{5}{16}\cos(3x) + \frac{5}{8}\cos(x).$$

Exercise 2

a) Use the De Moivre's theorem to prove that:

$$\sin(5\theta) = \sin(\theta)(16\cos^4(\theta) - 12\cos^2(\theta) + 1)$$

b) Solve the equation $16x^4 - 12x^2 + 1 = 0$, and determine the value of $\cos(\frac{\pi}{5})$.

Solution:

a) We have from the lessons:

$$\sin(n\theta) = \sum_{i=1}^{[\frac{n+1}{2}]} (-1)^{(i-1)} C_n^{(2i-1)} \cos^{(n-2i+1)}(\theta) \sin^{(2i-1)}(\theta)$$

So, for $n=5$, we get:

$$\sin(5\theta) = \sum_{i=1}^{3} (-1)^{i-1} C_5^{(2i-1)} \cos^{(6-2i)}(\theta) \sin^{(2i-1)}(\theta)$$

$$= C_5^1 \cos^4(\theta)\sin\theta - C_5^3 \cos^2(\theta)\sin^3(\theta) + C_5^5 \sin^5(\theta)$$

We calculate values of C_k^1, C_5^3, C_5^5 from Pascal's triangle:

then: $\sin(5\theta) = 5\cos^4(\theta)\sin\theta - 10\cos^2(\theta)(1-\cos^2(\theta))\sin\theta + (1-\cos^2(\theta))^2\sin(\theta)$

therefore:

$$\sin(5\theta) = \sin(\theta)(16\cos^4(\theta) - 12\cos^2(\theta) - 1) \quad \ldots\ldots (1)$$

b) Solve the eqn. $16x^4 - 12x^2 + 1 = 0$?

If we let $y = 2x$, we obtain: $y^2 = 4x^2$ and $y^4 = 16x^4$.

Now, substituting $y=2x$ into the last eqn. (1), we get:

$$y^4 - 3y^2 + 1 = 0 \Rightarrow y^4 + (y^3 - y^3) + (y - y) - 3y^2 + 1 = 0$$

$$\Rightarrow y^4 + y^3 - y^2 - y^3 - y^2 + y - y^2 - y + 1 = 0$$

$$\Rightarrow y^2(y^2 + y - 1) - y(y^2 + y - 1) - (y^2 + y - 1) = 0$$

$$\Rightarrow (y^2 + y - 1)(y^2 - y - 1) = 0$$

Therefore: $\begin{cases} y^2+y-1=0 \Rightarrow y = \dfrac{-1\pm\sqrt{5}}{2} \\ y^2-y-1=0 \Rightarrow y=\dfrac{1\pm\sqrt{5}}{2} \end{cases}$

finally, the solution $x \in \left\{ \dfrac{-1-\sqrt{5}}{4}, \dfrac{-1+\sqrt{5}}{4}, \dfrac{1-\sqrt{5}}{4}, \dfrac{1+\sqrt{5}}{4} \right\}$

• Determine the value of $\cos(\frac{\pi}{5})$?

From question ⓐ in exercise 2, we have this eqn:

$$\cos(5\theta) = \sin(\theta)\left(16\cos^4(\theta) - 12\cos^2(\theta) - 1\right) \quad \dots \ (1)$$

let us substitute $\theta = \frac{\pi}{5}$ in eqn (1) to obtain:

$$\cos(5\tfrac{\pi}{5}) = \sin(\tfrac{\pi}{5})\left(16\cos^4(\tfrac{\pi}{5}) - 12\cos^2(\tfrac{\pi}{5}) - 1\right)$$

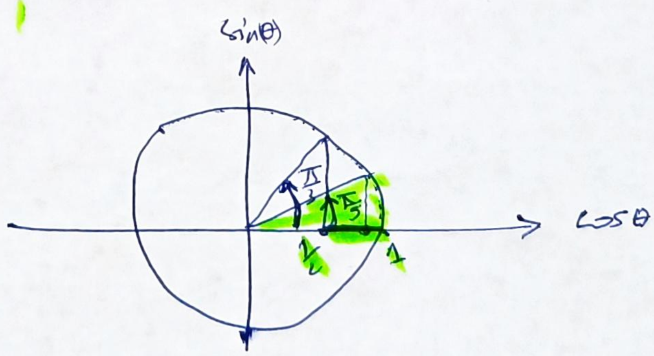
$$\Rightarrow \quad 0 = \sin(\tfrac{\pi}{5})\left(16\cos^4(\tfrac{\pi}{5}) - 12\cos^2(\tfrac{\pi}{5}) - 1\right) = 0$$

$$\Rightarrow 16\cos^4(\tfrac{\pi}{5}) - 12\cos^2(\tfrac{\pi}{5}) - 1 = 0 \quad \dots \ (2)$$

from the last equality (2), we can deduce that $\cos(\frac{\pi}{5})$ is

a solution, so $\cos(\frac{\pi}{4}) = \frac{1}{4} + \frac{1}{4}\sqrt{5}$

becouse $0 < \frac{\pi}{5} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos\frac{\pi}{5} < 1$.



Exercice 3

- By using De Moivre's theorem, prove that:

$$S = 1+\cos(\theta)+\cos(2\theta)+\cdots+\cos(n-1)\theta = \frac{\sin(n-\frac{1}{2})\theta}{2\sin(\frac{\theta}{2})}+\frac{1}{2} \; ; \forall \theta \neq 2\pi k,$$

$k\in\mathbb{Z}, n\in\mathbb{N}^*$.

Solution:

We know that:

$$S = 1+\cos(\theta)+\cos(2\theta)+\cdots+\cos(n-1)\theta = Re\{1+e^{i\theta}+\cdots+e^{i(n-1)\theta}\}.$$

If $\theta\neq 2\pi k$, $k\in\mathbb{Z}^*$, then $e^{i\theta}\neq 1$, hence according to the formula for the summation of geometric sequence of the common ratio $r=e^{i\theta}$, we have:

$$A = 1+e^{i\theta}+e^{i2\theta}+\cdots+e^{i(n-1)\theta} = \frac{e^{in\theta}-1}{e^{i\theta}-1}$$

Now, since S is the real part of A; that is:

$$S = 1+\cos(\theta)+\cos(2\theta)+\cdots+\cos(n-1)\theta = Re\left\{\frac{e^{in\theta}-1}{e^{i\theta}-1}\right\}$$

$$= Re\left\{\frac{(e^{in\theta}-1)}{(e^{i\theta}-1)}\cdot\frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}\right\} = Re\left\{\frac{e^{i(n-\frac{1}{2})\theta}-e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}\right\}$$

Since $\sin(\frac{\theta}{2}) = \frac{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}{2i}$, $e^{i(n-\frac{1}{2})\theta} = \cos(n-\frac{1}{2})\theta+i\sin(n-\frac{1}{2})\theta$

and ~~$\sin(\frac{\theta}{2})$~~ $e^{-i\frac{\theta}{2}} = \cos(\frac{\theta}{2})-i\sin(\frac{\theta}{2})$, we obtain:

$$S = Re\left\{\frac{\cos(n-\frac{1}{2})\theta+i\sin(n-\frac{1}{2})\theta-\cos(\frac{\theta}{2})+i\sin(\frac{\theta}{2})}{2i\sin(\frac{\theta}{2})}\right\}$$

$$= \frac{\sin(n-\frac{1}{2})\theta}{2\sin(\frac{\theta}{2})}+\frac{1}{2}$$

Summary of Annotations