3 Real sequences

3.1 Generalities

Definition 3.1

· We call each fonction U of $\mathbb{N}$ in $\mathbb{R}$; a real sequence.

$$\begin{aligned} U &: \mathbb{N} \longrightarrow \mathbb{R} \\ n &\rightarrow u_n = u(n) \end{aligned}$$

· u_n is called the general term of the sequence U.
· We olso sumbolize the sequence by $(u_n)_{n\in\mathbb{N}}$ or (u_n).
· If the sequence is defined for each $n \geq n_0$ we denote it by $(u_n)_{n\geq n_0}$.
· A real sequence is defined explicitly or with a recurrent relation.

examples

1) $(u_n)_{n\geq 2}$ is a sequence defined by its general term:

$$\forall n \geq 2 : u_n = \sqrt{n-2}.$$

So

$$u_2 = 0; u_3 = 1; u_4 = \sqrt{2}; u_5 = \sqrt{3}; ....; u_{10} = \sqrt{8}.....$$

2) $(v_n)_{n\in\mathbb{N}}$ is a sequence defined by a recurrent relation:

$$\forall n \in \mathbb{N} : v_{n+1} = \frac{v_n}{v_n+1} \quad ; \quad v_0 = 1 \quad .$$

So

$$\begin{aligned} v_1 &= \frac{v_0}{v_0+1} = \frac{1}{1+1} = \frac{1}{2}; \\ v_2 &= \frac{v_1}{v_1+1} = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}; \\ v_3 &= \frac{v_2}{v_2+1} = \frac{\frac{1}{3}}{\frac{1}{3}+1} = \frac{1}{4} \\ & \text{.................} \end{aligned}$$

prove that: $\forall n \in \mathbb{N} : v_n = \frac{1}{n+1}$.

Definition 3.2 Let (u_n) be a real sequence.

· (u_n) is bounded from **above** if and only if :

$$\exists M \in \mathbb{R}; \forall n \in \mathbb{N} : u_n \leq M.$$

· (u_n) is bounded from **below** if and only if :

$$\exists m \in \mathbb{R}; \forall n \in \mathbb{N} : u_n \geq m.$$

· (u_n) is bounded if and only if it is bounded from above and below.
In other words: $((u_n)$ is bounded$) \Leftrightarrow \left(\exists M \in \mathbb{R}_+^*; \forall n \in \mathbb{N} : |u_n| \leq M\right)$.

Definition 3.3 Let (u_n) be a real sequence.

· (u_n) is **increasing** (Strictly increasing, respectively) if and only if :

$$\forall n \in \mathbb{N} : u_n \leq u_{n+1} \quad (\forall n \in \mathbb{N} : u_n < u_{n+1}, \text{respectively }).$$

· (u_n) is **decreasing** (Strictly decreasing, respectively) if and only if :

$$\forall n \in \mathbb{N} : u_n \geq u_{n+1} \quad (\forall n \in \mathbb{N} : u_n > u_{n+1}, \text{respectively }).$$

· (u_n) is **canstant** if and only if :

$$\forall n \in \mathbb{N} : u_n = u_{n+1}.$$

· A sequence of real numbers (u_n) is called **monotone** if it is either increasing or decreasing.

3.2 Convergent sequences

Definition 3.4

A sequence (u_n) is convergent and its limit is the real number ℓ, if and only if:

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} \ [n > N \Longrightarrow |u_n - \ell| < \varepsilon].$$

and we write: $\lim\limits_{n \to \infty} u_n = \ell$ or $\lim\limits_{n} u_n = \ell$.

If (u_n) does not converge, then we say that it divergent.

Example

Let it be the sequence (u_n) where: $\forall n \in \mathbb{N} : u_n = \frac{3n+1}{n+2}$ and let's prove that $\lim\limits_{n} u_n = 3$.

Let $\varepsilon \in \mathbb{R}_+^*$ where $|u_n - \ell| < \varepsilon$, so

$$|u_n - \ell| < \varepsilon \Longleftrightarrow \left| \frac{3n+1}{n+2} - 3 \right| < \varepsilon$$

$$\Longleftrightarrow n > \frac{5}{\varepsilon} - 2.$$

According to Archimedean Axiom we have: $\exists N_0 \in \mathbb{N}$; $N_0 > \frac{5}{\varepsilon} - 2$, so we take $N = N_0$ until the following is achieved:

$$\forall \varepsilon > 0; \exists N \, (N = N_0) \in \mathbb{N}; \forall n \in \mathbb{N} \ [n > N \Longrightarrow |u_n - 3| < \varepsilon].$$

Remark: N can be determined in another way.

We have $\frac{5}{\varepsilon} - 2 \leq \left| \frac{5}{\varepsilon} - 2 \right|$ and $\left| \frac{5}{\varepsilon} - 2 \right| < E\left(\left| \frac{5}{\varepsilon} - 2 \right| \right) + 1 \in \mathbb{N}$, so it is enough we take $N = E\left(\left| \frac{5}{\varepsilon} - 2 \right| \right) + 1$.

Teorem 3.1 (Uniqueness of limit)

Let (u_n) be a convergent sequence,then the limit is unique.

Proof

Assume that the sequence (u_n) accepts two limits ℓ and ℓ' where $\ell' \neq \ell$.

For $\varepsilon = \frac{\left| \ell - \ell' \right|}{2}$, we have:

$$\exists N_1 \in \mathbb{N}; \forall n \in \mathbb{N} \ [n > N_1 \Longrightarrow |u_n - \ell| < \varepsilon] \ \text{and} \ \exists N_2 \in \mathbb{N}; \forall n \in \mathbb{N} \ \left[n > N_2 \Longrightarrow \left| u_n - \ell' \right| < \varepsilon \right]$$

by putting $N = \max\{N_1, N_2\}$; then $\forall n \in \mathbb{N}$;

$$\begin{aligned} n > N \Longrightarrow \left|\ell - \ell^{'}\right| &= \left|\ell - u_n + u_n - \ell^{'}\right| \\ &\leq \underbrace{|u_n - \ell|}_{<\varepsilon} + \underbrace{|u_n - \ell|}_{<\varepsilon} \\ &< 2\varepsilon = \left|\ell - \ell^{'}\right|. \end{aligned}$$

This is a contradiction, So $\ell^{'} = \ell$.

Teorem 3.2

Let (u_n) be a convergent sequence, then (u_n) is bounded.

Proof

We assume that the sequence (u_n) is convergent to towards the number ℓ.
For $\varepsilon = 1; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow \ell - 1 < u_n < \ell + 1$
by putting $A = \{u_0, u_1, ..., u_N, \ell - 1, \ell + 1\}$ then:

$$\forall n \in \mathbb{N} : \min A \leq u_n \leq \max A.$$

Teorem 3.3 Let (u_n) be a real sequence.

(i) If (u_n) is increasing and bounded from above, then (u_n) converges, where $\lim_n u_n = \sup_{n\in\mathbb{N}} u_n$.

(ii) If (u_n) is decreasing and bounded from below, then (u_n) converges, where $\lim_n u_n = \inf_{n\in\mathbb{N}} u_n$.

Remark So, a bounded monotone sequence converges.

Proof

(i) Let the sequence (u_n) be increasing and bounded from above.
The set $A = \{u_n; n \in \mathbb{N}\}$ is bounded from above. We put $\sup A = \ell$,we have:

$$\sup A = \ell \Longleftrightarrow \begin{cases} \forall n \in \mathbb{N} : u_n \leq \ell \\ \forall \varepsilon > 0; \exists N \in \mathbb{N} : \ell - \varepsilon < u_N \end{cases}$$

so, $\forall n \in \mathbb{N}$:

$$n > N \Longrightarrow u_n \geq u_N \quad (\quad (u_n) \text{ is increasing })$$

$$\begin{aligned} &\Longrightarrow \ell \geq u_n \geq u_N > \ell - \varepsilon \\ &\Longrightarrow \ell + \varepsilon > u_n > \ell - \varepsilon \\ &\Longrightarrow |u_n - \ell| < \varepsilon. \end{aligned}$$

Hence

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow |u_n - \ell| < \varepsilon$$

we obtain

$$\lim_n u_n = \ell.$$

(ii) In the same way, we prove the second case.

Example

Let it be the sequence (u_n) defined by: $u_0 = \alpha > 1$ and $\forall n \in \mathbb{N} : u_{n+1} = \frac{2u_n+1}{u_n+2}$.

Prove that (u_n) is convergent (calculating the limit is not required).

We use proof by induction, to prove that: $\forall n \in \mathbb{N} : u_n > 1$, and we prove that (u_n) decreasing (strictly decreasing),according the theorem **3.3,** the sequence (u_n) is convergent.

Teorem 3.4

If the sequences (u_n) and (v_n) are converges towards ℓ and $\ell^{'}$ respectively then the sequences: $(u_n + v_n)$; $(u_n v_n)$; (λu_n) ; $(|u_n|)$ are converges towards $\ell + \ell^{'}$; $\ell\ell^{'}$; $\lambda\ell$; $|\ell|$ respectively. If $\ell^{'} \neq 0$ and $\forall n \in \mathbb{N} : v_n \neq 0$ then $\left(\frac{u_n}{v_n}\right)$ is converges towards $\frac{\ell}{\ell\prime}$.

Proof (Let us prove the last case)

We have $\lim\limits_{n} v_n = \ell^{'} \neq 0$. For $\varepsilon = \frac{\left|\ell^{'}\right|}{2}$, then $\exists N_1 \in \mathbb{N}$; $\forall n \in \mathbb{N}$:

$$n > N_1 \Longrightarrow \left|v_n - \ell^{'}\right| < \frac{\left|\ell^{'}\right|}{2}$$

$$\begin{aligned}
&\Longrightarrow \quad \left||v_n| - \left|\ell^{'}\right|\right| < \frac{\left|\ell^{'}\right|}{2} \text{ (Since } \left||v_n| - \left|\ell^{'}\right|\right| \leq |v_n - \ell'|) \\
&\Longrightarrow \quad \frac{1}{2}\left|\ell^{'}\right| < |v_n| < \frac{3}{2}\left|\ell^{'}\right| \\
&\Longrightarrow \quad \frac{1}{|v_n|} < \frac{2}{|\ell'|}.
\end{aligned}$$

On the other hand for $\varepsilon > 0$,then:

$$\exists N_2 \in \mathbb{N}; \forall n \in \mathbb{N} : n > N_1 \Longrightarrow |u_n - \ell| < \varepsilon$$

and

$$\exists N_3 \in \mathbb{N}; \forall n \in \mathbb{N} : n > N_3 \Longrightarrow \left|v_n - \ell^{'}\right| < \varepsilon$$

by putting: $N = \max\{N_1, N_2, N_3\}$, then:$\forall n \in \mathbb{N}$:

$$n > N \Longrightarrow \left|\frac{u_n}{v_n} - \frac{\ell}{\ell\prime}\right| = \left|\frac{u_n\ell^{'} - \ell v_n}{v_n\ell^{'}}\right| = \left|\frac{u_n\ell^{'} - \ell\ell^{'} + \ell\ell^{'} - \ell v_n}{v_n\ell^{'}}\right|$$

$$\begin{aligned}
&\leq \quad \frac{\left|(u_n - \ell)\,\ell^{'}\right| + \left|\ell\left(v_n - \ell^{'}\right)\right|}{|v_n\ell'|} \\
&< \quad \frac{2\left(|\ell| + \left|\ell^{'}\right|\right)}{|\ell'|^2}\varepsilon = \varepsilon^{'} \text{ (} \varepsilon > 0 \text{)}
\end{aligned}$$

so

$$\forall \varepsilon' > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow \left| \frac{u_n}{v_n} - \frac{\ell}{\ell'} \right| < \varepsilon'.$$

Teorem 3.5

· If (u_n) and (v_n) are sequences that converge towards ℓ and ℓ' respectively, where: $\forall n \in \mathbb{N} : u_n < v_n$ Then $\ell \leq \ell'$.

· If (u_n) , (v_n) and (w_n) are a convergent sequences verified: $\forall n \in \mathbb{N} : w_n < u_n < v_n$ and $\lim\limits_n v_n = \lim\limits_n w_n = \ell$,Then $\lim\limits_n u_n = \ell$

Proof

· Let (u_n) and (v_n) are a convergent sequences, where $\forall n \in \mathbb{N} : u_n < v_n$ with $\lim\limits_n u_n = \ell$ and $\lim\limits_n v_n = \ell'$

assume that $\ell > \ell'$ for $\varepsilon = \frac{\ell - \ell'}{2}$ then $\exists N_0 \in \mathbb{N}; \exists N_1 \in \mathbb{N}; \forall n \in \mathbb{N}$:

$$n > N_0 \Longrightarrow |u_n - \ell| < \varepsilon \Longrightarrow \ell - \varepsilon < u_n < \ell + \varepsilon \Longrightarrow \frac{\ell + \ell'}{2} < u_n < \frac{3\ell - \ell'}{2}$$

$$n > N_1 \Longrightarrow \left| v_n - \ell' \right| < \varepsilon \Longrightarrow \ell' - \varepsilon < v_n < \ell' + \varepsilon \Longrightarrow \frac{\ell + 3\ell'}{2} < v_n < \frac{\ell + \ell'}{2}$$

by putting $N = \max\{N_0, N_1\}$, then $\forall n \in \mathbb{N}$:

$$n > N \Longrightarrow v_n < \frac{\ell + \ell'}{2} < u_n.$$

This contradicts the hypothesis, $\forall n \in \mathbb{N} : u_n < v_n$.

· The second case is a result of first case.

3.3 Subsequences

Definition 3.5

Let (u_n) be a sequence. A subsequence (v_k) of the sequence (u_n) is defined by a function $f : \mathbb{N} \to \mathbb{N}$ such that f is strictly increasing, and $v_k = u_{f(k)}$ for $k \in \mathbb{N}$.

We often write n_k instead of $f(k)$.

Example

Let it be the sequence (u_n) defined by $\forall n \in \mathbb{N} : u_n = \frac{n}{n+1}$

· For $n_k = f(k) = 3k$ (f is strictly increasing) the subsequence (v_k) (or (u_{n_k})) defined by: $\forall k \in \mathbb{N} : v_k = u_{3k} = \frac{3k}{3k+1}$.

· For $n_k = f(k) = k^2 + 1$ (f is strictly increasing) the subsequence (w_k) defined by: $\forall k \in \mathbb{N} : w_k = u_{k^2+1} = \frac{k^2+1}{k^2+2}$.

	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	....
u_n	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	...
v_k	v_0			v_1			v_2			v_3		...
w_k		w_0	w_1			w_2					w_3	...

Teorem 3.6

Every subsequence of a convergent sequence is a convergent sequence and has the same limit.

To prove the previous theorem we need the folowing proposition

proposition 3.1

If (n_k) is a sequence of strictly increasing natural numbers, then $\forall k \in \mathbb{N}$: $n_k \geq k$.

Proof of proposition 3.1

For $k = 0$ we have $n_0 \geq 0$ (is true because $n_0 \in \mathbb{N}$).

Assume that $\forall k \in \mathbb{N} : n_k \geq k$.

We have $n_{k+1} > n_k$ (because (n_k) strictly increasing)

so $n_{k+1} > n_k \Longrightarrow n_{k+1} > k \Longrightarrow n_{k+1} \geq k + 1$.

Proof of theorem 3.6

Let (u_n) are a convergent sequence towards ℓ and (n_k) is a sequence of strictly increasing natural numbers, we have to prove that:$\lim_k u_{n_k} = \ell$.

We have $\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow |u_n - \ell| < \varepsilon$

so

$$\forall k \in \mathbb{N}, k > N \Longrightarrow n_k > n_N \quad (\ (n_k) \text{ is strictly increasing })$$

$$\begin{aligned} &\Longrightarrow \quad n_k > n_N \geq N \quad (\text{ using proposition 3.1 }) \\ &\Longrightarrow \quad |u_{n_k} - \ell| < \varepsilon. \end{aligned}$$

From this we conclude that the subsequence $(u_{n_k})_k$ is converges towards ℓ.

Remak Using the contrapositive form of implication in Theorem (3.6), we can prove the divergence of some sequences.

Example Let the sequence (u_n) defined by $\forall n \in \mathbb{N} : u_n = \frac{n+1}{n+2} \sin \frac{n\pi}{2}$

Let us construct the two subsequences $(u_{2k})_{k\in\mathbb{N}}$ and $(u_{4k+1})_{k\in\mathbb{N}}$, where $\forall k \in \mathbb{N} : u_{2k} = 0$ and $u_{4k+1} = \frac{4k+2}{4k+3}$.

We have $\lim_k u_{2k} = 0$ and $\lim_k u_{4k+1} = 1$, since $\lim_k u_{2k} \neq \lim_k u_{4k+1}$,so the sequence (u_n) is divergent.

3.4 Infinite limits

Definition 3.6

We say (u_n) diverges to infinity if and only if

$$\forall A \in \mathbb{R}; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow u_n > A.$$

In this case we write $\lim_n u_n = \infty$.

Similarly, we say (u_n) diverges to minus infinity and we write $\lim_n u_n = -\infty$ if and only if

$$\forall A \in \mathbb{R}; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow u_n < A.$$

Proposition 3.2 Suppose (u_n) is a monotone unbounded sequence. Then

$$\lim_n u_n = \begin{cases} \infty \text{ if } (u_n) \text{ is increasing,} \\ -\infty \text{ if } (u_n) \text{ is decreasing.} \end{cases}$$

Proof (Exercise**).**

3.5 Adjacent sequences

Definition 3.7

We say of two sequences (u_n) and (v_n) that they are adjacent if and only if one of them is decreasing and the other is increasing and $\lim_n (u_n - v_n) = 0$.

Theorem 3.7

Every two adjacent sequences are convergent sequences and have the same limit.

Proof

Let (u_n) and (v_n) be two adjacent sequences, where (u_n) is increasing and (v_n) is decreasing.

The sequence $v_n - u_n$ is decreasing, so it converges towards its infimum 0, from which $\forall n \in \mathbb{N} : \ v_n - u_n \geq 0$, or $\forall n \in \mathbb{N} : \ u_n \leq v_n$, so $\forall n \in \mathbb{N}$: $u_0 \leq \ u_n \leq v_n \leq v_0$.

So the sequences (u_n) and (v_n) are monotonic and bounded, so they are convergent.

Assume that $\lim_n u_n = l$ and $\lim_n v_n = l^{'}$, we have $\lim_n (u_n - v_n) = 0$, so $l - l^{'} = 0$, from which $l = l^{'}$.

Example Let the sequences (u_n) and (v_n) defined by:$\forall n \in \mathbb{N} : u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $v_n = u_n + \frac{1}{n}$.

Prove that (u_n) and (v_n) be two adjacent sequences.

We have $\forall n \in \mathbb{N} : u_{n+1} - u_n = \frac{1}{(n+1)^2} > 0$, (u_n) is strictly increasing.

$\forall n \in \mathbb{N} : v_{n+1} - v_n = -\frac{1}{n(n+1)^2} < 0$, (v_n) is strictly decreasing.

$\lim_n (u_n - v_n) = \lim_n \frac{-1}{n} = 0$

Thus the sequences (u_n) and (v_n) are adjacent, they are convergent and have the same limit (It can be proven that $\lim_n u_n = \lim_n v_n = \frac{\pi^2}{6}$).

Theorem 3.8 (BOLZANO-WEIERSTRASS)

From each bounded real sequence, a convergent subsequence can be extracted.

Proof Let (u_n) be a bounded sequence, we put $a_0 = \inf_{n\in\mathbb{N}} u_n$ and $b_0 = \sup_{n\in\mathbb{N}} u_n$.

We have $\forall n \in \mathbb{N} : a_0 \leq u_n \leq b_0$, we put $I_0 = [a_0, b_0]$.

Let us divide the interval I_0 into two intervals of equal length. At least one of these two intervals contains an infinite number of terms of the sequence (u_n), which we denote by $I_1 = [a_1, b_1]$, and let u_{n_1} be one of the terms of the sequence (u_n), where $u_{n_1} \in I_1$.

Let us divide the interval I_1 into two intervals of equal length. At least one of these two intervals contains an infinite number of terms of the sequence (u_n), which we denote by $I_2 = [a_2, b_2]$, and let u_{n_2} be one of the terms of the sequence (u_n), where $u_{n_2} \in I_2$ and $n_2 > n_1$ (this is possible because I_2 contains an infinite number of terms of the sequence (u_n)).

Thus, we create a sequence of intervals $I_k = [a_k, b_k]$ where I_k is one of the two halves of the interval I_{k-1} which contains an infinite number of terms of the sequence (u_n) and u_{n_k} is one of the terms of the sequence (u_n) where $u_{n_k} \in I_k$ and $n_k > n_{k-1}$, then we get a subsequence (u_{n_k}) of the sequence (u_n) satisfies $\forall k \in \mathbb{N} : a_k \leq u_{n_k} \leq b_k$.

We have $\lim\limits_{k} (b_k - a_k) = \lim\limits_{k} \left(\frac{b_0 - a_0}{2^k}\right) = 0$ and since $I_k \subset I_{k-1}$ the sequence (a_k) is increasing and the sequence (b_k) is decreasing, so the sequences (a_k) and (b_k) are adjacent, and therefore the sequence (u_{n_k}) is convergent and its limit is the common limit of the sequences (a_k) and (b_k).

3.6 Cauchy sequence

Definition 3.8

Let (u_n) be a sequence. We say that (u_n) is a Cauchy sequence if

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall p, q \in \mathbb{N}; (p > N \wedge q > N) \Longrightarrow |u_p - u_q| < \varepsilon$$

Second formula

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall p, n \in \mathbb{N}; n > N \Longrightarrow |u_{n+p} - u_n| < \varepsilon$$

Theorem 3.9

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof

Necessary condition Let (u_n) be a sequence that converges to the real number l.

We have $\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow |u_n - \ell| < \frac{\varepsilon}{2}$

so $\forall p, q \in \mathbb{N}$:

$$(p > N \wedge q > N) \Longrightarrow |u_p - u_q| = |u_p - l + l - u_q|$$

$$\begin{aligned} &\leq |u_p - l| + |l - u_q| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

so, (u_n) is a Cauchy sequence.

Sufficient condition Assume that (u_n) it is a Cauchy sequence

First: For $\varepsilon = 1$, then

$$\exists N_0 \in \mathbb{N}; \forall n, q \in \mathbb{N}; (n > N_0 \wedge q > N_0) \Longrightarrow |u_n - u_q| < 1.$$

And for $q = N_0 + 1$, then:

$$\forall n \in \mathbb{N}; n > N_0 \Longrightarrow |u_n - u_{N_0+1}| < 1$$

$$\begin{aligned} &\Longrightarrow ||u_n| - |u_{N_0+1}|| < 1 \\ &\Longrightarrow |u_n| < |u_{N_0+1}| + 1 \end{aligned}$$

So, the set $A = \{|u_0|, |u_1|, |u\_2|, \ldots\ldots, |u_{N_0}|, |u_{N_0+1}| + 1\}$ is finite as it accepts a maximum, we denote it by M, then

$$\forall n \in \mathbb{N} : |u_n| < M$$

So (u_n) is bounded.

Second: Since (u_n) bounded, according to **Theorem 3.8**, it is possible to extract from the sequence (u_n) a subsequence (u_{n_k}) that converges towards real number l.

Let $\varepsilon > 0$ then

$$\exists k_0 \in \mathbb{N}; \forall k \in \mathbb{N} : k > k_0 \Longrightarrow |u_{n_k} - l| < \frac{\varepsilon}{2}$$

and

$$\exists N_1 \in \mathbb{N}; \forall p; q \in \mathbb{N} : (p > N_1 \wedge q > N_1) \Longrightarrow |u_p - u_q| < \frac{\varepsilon}{2}$$

by putting $N = \max\{k_0, N_1\}$, then $\forall p \in \mathbb{N}$:

$$p > N \Rightarrow p > k_0 \Longrightarrow |u_{n_p} - l| < \frac{\varepsilon}{2}$$

$$\begin{aligned} p > N \Longrightarrow p > N_1 &\Longrightarrow n_p \geq p > N_1 \quad (\text{ since proposition 3.1 }) \\ &\Longrightarrow |u_p - u_{n_p}| < \frac{\varepsilon}{2} \end{aligned}$$

so, $\forall p \in \mathbb{N}$:

$$\begin{aligned} p > N \Longrightarrow |u_p - l| &= \left|u_p - u_{n_p} + u_{n_p} - l\right| \\ &\leq \left|u_p - u_{n_p}\right| + \left|u_{n_p} - l\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall p \in \mathbb{N} : p > N \Longrightarrow |u_p - l| < \varepsilon.$$

So the sequence (u_n) is convergent towards l.

Remarks

1) One reason this is so useful is that it gives us a way to show that a sequence converges without needing to know in advance what the limit is.

2) A sequence (u_n) is divergent if and only if

$$\exists \varepsilon > 0; \forall N \in \mathbb{N}; \exists p; q \in \mathbb{N} : p > N \wedge q > N \wedge |u_p - u_q| \geq \varepsilon.$$

Example 1 Let the sequence (u_n) be defined by:$\forall n \in \mathbb{N}^* : u_n = \sum_{k=1}^{n} \frac{1}{k}$. Let us prove that (u_n) is divergent. Indeed

We have $\forall n \in \mathbb{N}^*$:

$$\begin{aligned}
u_{2n} - u_n &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \\
&= \sum_{k=n+1}^{2n} \frac{1}{k} \\
&\geq \sum_{k=n+1}^{2n} \frac{1}{2n} \text{ (because } \forall p \in \mathbb{N} : 1 \leq p \leq n \Longrightarrow \frac{1}{n+p} \geq \frac{1}{2n} \text{)} \\
&\geq n\frac{1}{2n} = \frac{1}{2}.
\end{aligned}$$

By putting $q = n$, $p = 2n$, $\varepsilon = \frac{1}{2}$, the following is achieved

$$\exists \varepsilon > 0 \; (\varepsilon = \frac{1}{2}) \; ; \forall n \in \mathbb{N}; \exists p; q \in \mathbb{N} : p \geq n \wedge q \geq n \wedge |u_p - u_q| \geq \varepsilon.$$

So, (u_n) is divergent.

Example 2

Let (u_n) be a real sequence where: $\forall n \in \mathbb{N} : |u_{n+1} - u_n| \leq \left(\frac{1}{2}\right)^n$, Prove that (u_n) is a Cauchy sequence.

For $n; p \in \mathbb{N}$ then:

$$\begin{aligned}
|u_{n+p} - u_p| &= |u_{n+p} - u_{n+p-1} + u_{n+p-1} - u_{n+p-2} + u_{n+p-2} - u_{n+p-3} + ... + u_{n+1} - u_n| \\
&\leq |u_{n+p} - u_{n+p-1}| + |u_{n+p-1} - u_{n+p-2}| + |u_{n+p-2} - u_{n+p-3}| + ... + |u_{n+1} - u_n| \\
&\leq \left(\frac{1}{2}\right)^{n+p-1} + \left(\frac{1}{2}\right)^{n+p-2} + \left(\frac{1}{2}\right)^{n+p-3} + ... + \left(\frac{1}{2}\right)^{n} \\
&\leq \left(\frac{1}{2}\right)^{n} \left(\left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2}\right)^{p-2} + \left(\frac{1}{2}\right)^{p-3} + ... + 1\right) \\
&\leq \left(\frac{1}{2}\right)^{n} \frac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} = 2\left(\frac{1}{2}\right)^{n} \left(1 - \left(\frac{1}{2}\right)^{p}\right) \\
&\leq 2\left(\frac{1}{2}\right)^{n} \text{ (because } 1 - \left(\frac{1}{2}\right)^{p} \leq 1 \text{).}
\end{aligned}$$

Since $\lim\limits_{n} 2\left(\frac{1}{2}\right)^n = 0$, Then $\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n \in \mathbb{N} : n > N \Longrightarrow 2\left(\frac{1}{2}\right)^n < \varepsilon$.

So

$$\forall \varepsilon > 0; \exists N \in \mathbb{N}; \forall n; p \in \mathbb{N}; n > N \Longrightarrow |u_{n+p} - u_n| < \varepsilon.$$

3.7 Recurrence Sequences

Definition 3.9

Let $f : D \longrightarrow \mathbb{R}$ be a function, where $f(D) \subset D$ and $\alpha \in D$.We say that the sequence (u_n) is recurrente if it is defined by $u_0 = \alpha$ and the recurrente relation: $\forall n \in N : u_{n+1} = f(u_n)$.

Monotonicity The monotonicity of the sequence (u_n) is related to the monotonicity of the function f. Using proof by induction, the following can be proven true:

Proposition 3.3

(i) If f is increasing, the sequence (u_n) is monotonic, increasing if $f(u_0) - u_0 \geq 0$ and decreasing if $f(u_0) - u_0 \leq 0$.

(ii) If f is decreasing, the sign of the difference $u_{n+1} - u_n$ is alternately negative and positive, which means that (u_n) is non-monotonic in this case.

Proof

(i) Assume that f is increasing

for $f(u_0) - u_0 \geq 0$, let us prove that: $\forall n \in \mathbb{N} : u_{n+1} - u_n \geq 0$.

$u_1 - u_0 = f(u_0) - u_0 \geq 0$ (it is true)

suppose that $u_{n+1} - u_n \geq 0$ or $u_{n+1} \geq u_n$.

We have $u_{n+1} \geq u_n \Longrightarrow f(u_{n+1}) \geq f(u_n) \Longrightarrow u_{n+2} \geq u_{n+1}$.

In the same way, we prove that: if $f(u_0) - u_0 \leq 0$ Then: $\forall n \in \mathbb{N} : u_{n+1} - u_n \leq 0$.

(ii) Assume that f is decreasing

if $u_{n+1} - u_n \geq 0$ we have

$u_{n+1} - u_n \geq 0 \Longrightarrow u_{n+1} \geq u_n \Longrightarrow f(u_{n+1}) \leq f(u_n) \Longrightarrow u_{n+2} \leq u_{n+1} \Longrightarrow u_{n+2} - u_{n+1} \leq 0$

So, $u_{n+1} - u_n \geq 0$ and $u_{n+2} - u_{n+1} \leq 0$. That is, the sign of the difference $u_{n+1} - u_n$ is alternately negative and positive.

Convergence

Proposition 3.4 We assume that f is continuous on D. If the sequence (u_n) converges towards l in D, then l is a solution to the equation $f(x) = x$.

Proof

If the sequence (u_n) converges towards l of D then: $\lim_n u_n = l \Longrightarrow \lim_n u_{n+1} = l$.

Since f is continuous at l, then: $\lim_n f(u_n) = f(l)$.

On the other hand, we have: $\lim_n u_{n+1} = \lim_n f(u_n) \Longrightarrow l = f(l)$ so, l is a solution to the equation $x = f(x)$.

Remark

Searching for the limit of the sequence (u_n) leads to the solution of the equation $f(x) = x$ with the unknown x in set D. If the equation does not accept solutions, then the sequence does not accept a limit. However, If the equation accepts one or more solutions, then the problem returns to studying the possibility that one of these solutions is the limit of the sequence (u_n). If the equation $f(x) = x$ accepts solutions, this does not necessarily mean that the sequence (u_n) is convergent.

Examples

1) Let the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = a$ and $\forall n \in \mathbb{N} : u_{n+1} = \sqrt{u_n + 2}$. We put $f(x) = \sqrt{x+2}$.

Since the function f is defined, continuous and strictly increasing on the domain $D = [-2, +\infty[$ and $f(D) \subset D$, the sequence (u_n) is defined and monotonic. The direction of its change is determined by the sign of the difference $f(u_0) - u_0$.

We have

$$f(u_0) - u_0 = f(a) - a = \sqrt{a+2} - a = \frac{-a^2 + a + 2}{\sqrt{a+2} + a} = \frac{(1+a)(2-a)}{\sqrt{a+2}+a}.$$

So, the sign of $f(u_0) - u_0$ from the sign of $(2-a)$ and the equation $\sqrt{x+2} = x$, accepts a single solution, $x = 2$, from which the following results:

(i) If $a < 2$, the sequence is strictly increasing, and we can prove that $\forall n \in \mathbb{N} : u_n < 2$, So the sequence is bounded from above by 2.

(ii) If $a > 2$,the sequence is strictly decreasing and bounded from below by 2.

(iii) If $a = 2$,the sequence is constant.

So the sequence is convergent in all cases and its limit is 2.

2) Let be the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = a > 1$ and $\forall n \in \mathbb{N} : u_{n+1} = u_n^2$. We put $f(x) = x^2$

Since the function f is defined, continuous, and strictly increasing on the domain $D = [0, +\infty[$ and $f(D) \subset D$, where $f(a) - a = a^2 - a > 0$, the sequence (u_n) is defined and strictly increasing.The equation $f(x) = x$ accepts two solutions, $x = 0$; $x = 1$, but the sequence (u_n) is divergent because:

Using proof by induction, we prove that $\forall n \in \mathbb{N} : u_n = a^{2^n}$ and hence $\lim_n u_n = +\infty$.