Series of exercises 2 (*Questions marked * left to the students*)

Exercise 01 *Study the monotonicity of the following sequences.*

$u_n = \frac{3n+4}{2n-1}$ *, $v_n = \frac{\sqrt{n+1}}{2n}$, $w_n = \propto n + (-1)^n$ ($\propto \in \mathbb{R}$) , $k_n = \left(1+\frac{1}{n}\right)^n$, $f_n = \left(1+\frac{1}{n}\right)^{n+1}$*.

For the two sequences (k_n);(f_n) we use Bernoulli's inequality

$$\forall n \in \mathbb{N}; \forall a > -1: (1+a)^n \geq 1 + na$$

Exercise 02 *Calculate the limit of each following sequences*

$a_n = \frac{3^n+(-1)^n}{2^n+2(-1)^n}$, $b_n = \frac{1+3+\cdots+(2n-1)}{n^2+n}$, $c_n = \frac{1+a+a^2+\cdots+a^n}{1+b+b^2+\cdots+b^n}$ *($|a| < 1$; $|b| < 1$)

, $d_n = \left(1+\frac{1}{n}\right)^n$, $e_n = \sqrt[n]{n^p}$)* ($p \in \mathbb{N}^*$) , $f_n = \left(\frac{n+a}{n+1}\right)^n$*($a \in \mathbb{R}$) , $g_n = \frac{1+2+2^2+\cdots+2^n}{2^n}$.

Exercise 03

1) Using the restriction calculate the limit of each following sequence.

a)* $(u_n)_{n\in\mathbb{N}^*}: u_n = \sum_{k=1}^{n} \frac{n}{n^2+k}$ b) $(v_n)_{n\in\mathbb{N}^*}: v_n = \frac{1}{n^2}\sum_{k=1}^{n} E(kx)$ (where $x \in \mathbb{R}$).

c) $(w_n)_{n\in\mathbb{N}^*}: w_n = \frac{n!}{n^n}$.

2) Using the definition prove that:

a) $\lim\left(\sqrt{n+1}-\sqrt{n}\right) = 0$, b)* $\lim \frac{n^2-1}{2n^2+n} = \frac{1}{2}$, c) $\lim \frac{\sqrt{n^2+1}}{n} = 1$,

d) $\lim a^n = +\infty$ (*where* $a > 1$) e)* $\lim \frac{2n-1}{n+3} = 2$ f) $\lim \frac{2^n+(-1)^n}{2^n} = 1$.

3) Prove that the sequence (u_n) is divergent in each of the following cases:

a) $u_n = (-1)^n \frac{n+2}{n}$, b)* $u_n = sin\left(\frac{n^2+1}{4n}\pi\right)$

Exercise 04

Let be (u_n) a real sequence defined by .$\forall n \in \mathbb{N}: u_{n+1} = 1 + \frac{1}{u_n}$ *and* $u_0 = 1$.

1) *Prove that* $\forall n \in \mathbb{N} : u_n \geq 1$.

2) *We symbolize* a *for the positive solution of the equation* $x = 1 + \frac{1}{x}$.

a) *prove that* $\forall n \in \mathbb{N}: |u_{n+1} - a| \leq \frac{1}{a}|u_n - a|$ *and* $|u_n - a| \leq \frac{1}{a^n}|u_0 - a|$.

b) *what do you conclude?.*

Exercise 05

1) *Prove that if the two subsequence* (u_{2n+1}) *and* (u_{2n}) *are converges towards* ℓ, then the *sequence* (u_n) *is converges towards* ℓ

2) ***Application 1:*** *Let* (u_n) *be a sequence where:* $\forall p \in \mathbb{N}^*; \forall n \in \mathbb{N}^*: 0 \leq u_{n+p} \leq \frac{1}{n} + \frac{1}{p}$.

Prove that (u_n) *converges towards* 0.

3) ***Application 2:*** *Let* (v_n) *be a decreasing and converges sequence towards* 0 *and let the sequence* (S_n) *defined by* $S_n = \sum_{i=0}^{n}(-1)^i v_i$.

Prove that the two subsequences (S_{2n}) *and* (S_{2n+1}) *adjacent, what do you conclude ?*

(Application 2 is a proof of Leibniz's theorem for series.).

<u>Exercise 06</u> *Let* a , b *be real numbers, where* $0 < a < b$. we define the two sequences (u_n) and (v_n) as follows.

$$\forall n \in \mathbb{N} : u_{n+1} = \sqrt{u_n . v_n} \quad , \quad v_{n+1} = \frac{u_n + v_n}{2} \quad , \quad v_0 = b \text{ ، } u_0 = a.$$

Prove the following

1) $\forall n \in \mathbb{N} : 0 < u_n < v_n$ 2). *the two sequences* (u_n) *and* (v_n) *are monotonic.*

3) $\forall n \in \mathbb{N}: v_{n+1} - u_{n+1} \leq \frac{1}{2}(v_n - u_n)$ 4) $\forall n \in \mathbb{N}: v_n - u_n \leq \left(\frac{1}{2}\right)^n (b - a)$

5) $\lim_{n\to\infty} (v_n - u_n) = 0$, *what do you conclude?*

<u>Exercise 07</u>

1) Let (u_n) be a real sequence, where $\forall n \in \mathbb{N}: |u_{n+1} - u_n| \leq a^n$ $(0 < a < 1)$.

Prove that (u_n) *is a Cauchy sequence.*

2)* Let (v_n) be a real sequence, where $\forall n \in \mathbb{N}^*: |v_{n+1} - v_n| \leq K|v_n - v_{n-1}| (0 < K < 1)$.

Prove that (v_n) *is a Cauchy sequence*

<u>Exercise 08</u>* *Let* α *be a real number and* $(u_n)_{n\geq 1}$ *a real sequence defined by*

$$\forall n \in \mathbb{N}^*: u_{n+1} = \frac{n}{(n+1)^2} u_n + \frac{2(n^2+n+1)}{(n+1)^2}; u_1 = \alpha$$

1) a) *Prove that the sequence* (u_n) *is monotonic and bounded*

b) Calculate $\lim_{n\to\infty} u_n$ (*we denote this limit as* ℓ).

2) a) *Find a simple relation between* $u_{n+1} - \ell$ and $u_n - \ell$.

b) *In terms of* n *and* α *deduce the expression for* u_n.

<u>Exercise 09</u>*

1) *Let the two sequences* $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *where* $\forall n \in \mathbb{N}^*: v_n = \frac{u_1 + u_2 + \cdots + u_n}{n}$

Prove the following:

a) *If* (u_n) *is monotonic, then* (v_n) *is monotonic and has the same direction of change with* (u_n).

b) If (u_n) *converges towards* ℓ, *then* (v_n) *is converges towards* ℓ.

2) *Let* $a_1, a_2 \dots\dots, a_m$ *be a positive real numbers, that are not all zero where* $m \in \mathbb{N}^*$.

Prove: that $\lim\limits_{n\to\infty} (a_1{}^n + a_2{}^n + \cdots + a_m{}^n)^{\frac{1}{n}} = \max\limits_{1\le i\le m} a_i$.