Chapter one: The set of real numbers

1 1.Algebraic structure of the set $\mathbb{R}$

The set of real numbers is a set that we denote by $\mathbb{R}$ equipped with the operation of addition and multiplication and an overall ordering relationship$\leq$checki the following Axiom.

A1) $\forall x, y, z \in \mathbb{R}: x + (y + z) = (x + y) + z.$

A2) $\forall x, y \in \mathbb{R}: x + y = y + x.$

A3) $\forall x \in \mathbb{R}: x + 0 = 0 + x = x.$

A4) $\forall x \in \mathbb{R}: x + (-x) = (-x) + x = 0.$

A5) $\forall x, y, z \in \mathbb{R}: x \cdot (y \cdot z) = (x \cdot y) \cdot z.$

A6) $\forall x, y \in \mathbb{R}: x \cdot y = y \cdot x.$

A7) $\forall x \in \mathbb{R}: x \cdot 1 = 1 \cdot x = x.$

A8) $\forall x \in \mathbb{R}^*: x \cdot x^{-1} = x^{-1} \cdot x = 1.$

A9) $\forall x, y, z \in \mathbb{R}: x \cdot (y + z) = x \cdot y + x \cdot z.$

A10) $\forall x \in \mathbb{R}: x \leq x.$

A11) $\forall x, y, z \in \mathbb{R}: \left(x \leq y \text{ و } y \leq z\right) \Rightarrow (x \leq z).$

A12) $\forall x, y \in \mathbb{R}: \left(x \leq y \text{ و } y \leq x\right) \Rightarrow (x = y).$

A13) $\forall x, y \in \mathbb{R}: x \leq y \text{ أو } y \leq x.$

A14) $\forall x, y, z \in \mathbb{R}: (x \leq y) \Leftrightarrow (x + z \leq y + z).$

A15) $\begin{cases} \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_+^*: (x \leq y) \Leftrightarrow (x \cdot z \leq y \cdot z) \\ \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_-^*: (x \leq y) \Leftrightarrow (x \cdot z \geq y \cdot z) \end{cases}.$

Properties

1) $\forall x, y, x', y' \in \mathbb{R}: \left(x \leq y \text{و} x' \leq y'\right) \Rightarrow (x + x' \leq y + y').$

2) $\forall x, y, x', y' \in \mathbb{R}_+^*: \left(x \leq y \text{ و} x' \leq y'\right) \Rightarrow (x \cdot x' \leq y \cdot y').$

4) $\forall x, y, x', y' \in \mathbb{R}_+^*: (0 < x < y) \Rightarrow \left(0 < \frac{1}{y} < \frac{1}{x}\right).$

1.2 Absolute value

Definition 1.1 let it be $x \in \mathbb{R}$

The absolute value of the real number x is the positive real number which we denote by $|x|$and defined as

$$|x| = \begin{cases} x, si\ x \geq 0 \\ -x, si\ x \leq 0 \end{cases}$$

Properties : $x. y\ r.$ is a real number where $r \geq 0$

1) $|x| \geq 0$; $|-x| = |x|$; $-|x| \leq x \leq |x|$

2) $|x| = 0 \iff x = 0$

3) $|x.y| = |x||y|$

4) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $(y \neq 0)$

5) $|x + y| \leq |x| + |y|$

6) $|x| \leq r \iff -r \leq x \leq r$

7) .$|x| \geq r \iff x \leq -r$ or $x \geq r$

1.3.Limited parts from $\mathbb{R}$

Definition 1.2

Let A be a sub set of $\mathbb{R}$ and non-empty .

We say that A is bounded from above if and only if :

$$\exists b \in \mathbb{R} \,;\, \forall x \in A : x \leq b$$

We say that A is bounded from below if and only if

$$\exists a \in \mathbb{R} \,;\, \forall x \in A : x \geq a$$

A is bounded if and only if it is bounded from above and

Proposition 1.1 *The three following conditions are equivalent*

1).*A is bounded*

2) $\exists a \in \mathbb{R}$; $\exists b \in \mathbb{R} : \forall x \in A : a \leq x \leq b$.

3) $\exists M \in \mathbb{R}_+^*$; $\forall x \in A : |x| \leq M$

1.3.1 sup and inf.max and min

- *The smallest upper limit from A is called sup A*

The biggest lower limit from A is called inf A

If $supA \in A$ *it is called* $\max A$

If $infA \in A$ *it is called* $\min A$

Note

If A is infinite from above (from lowest, respectively) in $\mathbb{R}$ we write $supA = +\infty$

($infA = -\infty$, *respectively).*

proposition 1.2

1)Let A be bounded from above, then

$$M = sup\ A \Leftrightarrow \begin{cases} \forall x \in A : x \leq M \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists a \in A : M - \varepsilon < a \end{cases}$$

2) Let A be bounded from below, then

$$m = inf\ A \Leftrightarrow \begin{cases} \forall x \in A : x \geq m \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in A : m + \varepsilon > b \end{cases}$$

Proof

1) M is the smallest of the upper bounds if and only if the following proposition is false .

$$\exists M' < M; \forall x \in A : x \leq M'$$

Is true. So, if the proposition $\forall M' < M; \exists x \in A : x > M'$.

By putting $\varepsilon = M - M'$ $(\varepsilon > 0)$ so, the last proposition is written in the form:

$$\forall \varepsilon > 0\ ;\ \exists x \in A : M - \varepsilon < x.$$

2) In the same way we prove the second case

Examples

1) $A = [1,2[$; $maxA =$ unvailable ;supA =2 ; inf A=1 $minA = 1$

2) $A = \{\frac{1}{n}\ ; n \in \mathbb{N}^*\}$

.$sup\ A = max\ A = 1$ فإن*then*$1 \in A \forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow\ 0 < \frac{1}{n} \leq 1$

Let we proof that $infA = 0$

$$0 = infA \Leftrightarrow \begin{cases} \forall x \in A : x \geq 0 \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in A : 0 + \varepsilon > b \end{cases}$$

On the other side we have $\forall \varepsilon > 0\ ;\ \exists b \in A : 0 + \varepsilon > b \Leftrightarrow \forall \varepsilon > 0\ ;\ \exists n \in \mathbb{N}^* : \frac{1}{n} < \varepsilon$.

$and\ this\ last\ proposition\ is\ true\ and\ its\ according\ to\ archimed's\ axiom$

$$\forall \varepsilon > 0\ ;\ \exists n \in \mathbb{N}^* : \mathrm{n}\varepsilon > 1$$

$\min A = unvailable$, because $0 \notin \mathrm{A}$.

1.3.2 Axiom of supermum and infimum:

*Any non-empty subset A of the real's $\mathbb{R}$ which is bounded above has a **supermum** in $\mathbb{R}$.*
*Any non-empty subset A of the real's $\mathbb{R}$ which is bounded below has a **infimum** in $\mathbb{R}$.*

1.4 Archimedean axiom

Theorem 1.1:$\forall x > 0;\ \forall y \in \mathbb{R}\ ; \exists n \in \mathbb{N}^* : y < nx.$

Proof:

We suppose that:

$\exists x > 0;\ \exists y \in \mathbb{R}\,; \forall n \in \mathbb{N}^*: y \geq nx$ *or* $\exists x > 0;\ \exists y \in \mathbb{R}\,; \forall n \in \mathbb{N}^*: n \leq \frac{y}{x}$,

that's mean the set $\mathbb{N}^*$is limited from above it accepts an upper limit in $\mathbb{R}$ called M.

So $\forall \varepsilon > 0\ ;\ \exists n_0 \in \mathbb{N}^*\ :\ M - \varepsilon < n_0$

by putting $\varepsilon = 1, we\ get\ the\ following\ : \exists n_0 \in \mathbb{N}^*\ :\ M - 1 < n_0$

or $\exists n_0 \in \mathbb{N}^*\ :\ M < n_0 + 1$

but $n_0 + 1 \in \mathbb{N}^*$

this is a contradiction because $supA = M.$

1.5 The integer part of a real number

For every real number x there is only one integer which we denote as E(x) or $[x]$ *it achives*

$E(x) \leq x < E(x) + 1$

$E(x)$ *is called the integer part of the real x.*

In other words $E(x)$ is The largest integer less than or equal to x.

<u>Examples</u>

1) $E(0\,,1) = 0$ *since* $0 \leq 0{,}1 < 0 + 1.$
2) $E(-0\,,1) = -1$ *since*$-1 \leq -0{,}1 < -1 + 1.$
3) $\forall n \in \mathbb{N}^*: E\left(\frac{1}{n+1}\right) = 0$ *since* $\forall n \in \mathbb{N}^*: 0 \leq \frac{1}{n+1} < 0 + 1.$

1.6 dense groups in $\mathbb{R}$

Theorem 1.2 *between every two different real numbers there is at least one rational number.*

Proof

Let y and x be two real numbers where $x < y$.

According to Archimedean axiom $\exists n \in \mathbb{N}^*: 1 < n(y - x)$ *or* $nx + 1 < ny.$

On the other hand we have $E(nx) \leq nx < E(nx) + 1$ *or*

$$nx < E(nx) + 1 \leq nx + 1 < ny.$$

So $nx < E(nx) + 1 < ny$ *then* $x < \frac{E(nx)+1}{n} < y.$

Well the rational number$\frac{(nx)+1}{n}$ *is bounded between the two real numbers* $x, y.$

definition 1.3

we denote the set of irrational numbers with I or Q^c

Theorem 1.3_*between every two different real numbers there is at least one irrational number.*

To prove this theory we need the following two propositions

proposition *1.3* the number $\sqrt{2}$ *is an irrational number*

.**proposition** *1.4* if $x \in$ I and r ∈ Q ∗ then r$x \in$ I

Proof of the proposition 1.3

We suppose that $\sqrt{2} \in \mathbb{Q}$ *then there is only one duality of natural numbers (p,q) then* $\frac{p}{q} = \sqrt{2}$ *and* $gcd(p.q) = 1$*, then::*

we conclude that q^2 *divide* p^2 *since* q^2 *and* p^2 *prime* $\frac{p}{q} = \sqrt{2} \Leftrightarrow p = q\sqrt{2} \Leftrightarrow p^2 = 2q^2$ *among themselves then* q^2 *devide* 1 *thats mean* $q = 1$ *substitung in the previous equality we get* $p^2 = 2$ *and this is a contradiction because there is no natural number squared equal 2 .*

Proof of the proposition1.4 *We assume* $x \in I$ *and* $r \in \mathbb{Q}^*$ *and that* $rx \in \mathbb{Q}$ *and from him:*

$$\left(\frac{1}{r} \in \mathbb{Q}^* or\ rx \in \mathbb{Q}\right) \Rightarrow \frac{1}{r}rx \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$$

This is a contradiction because $x \in I$.

Proof of the theorem 1.3

Let y, x *be a real numbers, where* $x < y$*, according to the theorem 1.2, there sexist a rational number* r $(r \neq 0)$ such that: $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$ or $x < r\sqrt{2} < y$ and according to propositions 1.3 and 1.4 we conclude that $r\sqrt{2}$ is a irrational number.

Corollary 1.1 The two sets $\mathbb{Q}$ and I is dense in $\mathbb{R}$.

1.7 Intervals in $\mathbb{R}$

Let a, b *a real numbers, where* $a < b$*, we define*

$[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$ *is called closed interval.*

$]a, b[= \{x \in \mathbb{R}: a < x < b\}$ *is called open interval.*

$[a, b[= \{x \in \mathbb{R}: a \leq x < b\}$ *is called half open interval.*

$]a, b] = \{x \in \mathbb{R}: a < x \leq b\}$ "

$[a, +\infty[= \{x \in \mathbb{R}: x \geq a\}$ *unbounded closed interval.*

$]-\infty, b] = \{x \in \mathbb{R}: x \leq b\}$ "

$]a, +\infty[= \{x \in \mathbb{R}: x > a\}$ *unbounded open interval.*

$]-\infty, b[= \{x \in \mathbb{R}: x < b\}$ "

$\mathbb{R} =]-\infty, +\infty[$ "

Theorem 1.4

The non-empty subset I of $\mathbb{R}$ *is an interval if and only if the following property is satisfied:*

$$\forall a, b \in I\ (a \leq b);\ \forall x \in \mathbb{R}: a \leq x \leq b \Rightarrow x \in I$$

Proof

(⇐)Necessary condition: It is a clear that: if the set I is a interval, then the property is true.

(⇒)Sufficient condition: If the property is true, then the set I is a interval.

We have four possible cases, case 1: I is bounded, case 2: I is bounded from above and not bounded from below, case 3: I is bounded from below and not bounded from above, case 4: I is neither bounded from above nor from below.

Let us prove that in the first case: either $I = [a, b]$ or $I = [a, b[$ or $I =]a, b]$ or $I =]a, b[$ where $a = inf\ I$ and $b = sup\ I$.

We have: $$b = sup\ I \Leftrightarrow \begin{cases} \forall x \in I : x \leq b \\ \text{و} \\ \forall \varepsilon > 0\ ;\ \exists b' \in I : b - \varepsilon < b' \dots\dots (1) \end{cases}.$$

and

$$a = inf\ I \Leftrightarrow \begin{cases} \forall x \in I : x \geq a \\ \text{و} \\ \forall \delta > 0\ ;\ \exists a' \in I : a + \delta > a' \dots\dots (2) \end{cases}.$$

case 1: *If* $a \in I$ *and* $b \in I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x \leq b \Rightarrow x \in [a, b] \Rightarrow I \subset [a, b]$$

$$\forall x \in \mathbb{R}: x \in [a, b] \Rightarrow a \leq x \leq b \Rightarrow x \in I \Rightarrow [a, b] \subset I$$

So

$$I = [a, b].$$

case 2: *If* $a \in I$ *and* $b \notin I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x < b \Rightarrow x \in [a, b[\Rightarrow I \subset [a, b[$$

$$\forall x \in \mathbb{R}: x \in [a, b[\Rightarrow a \leq x < b \Rightarrow b - x > 0$$

putting $\varepsilon = b - x$ *in* (1) *we get* $x < b'$ *and since* $a, b' \in I$, *then:*

$$a \leq x < b' \Rightarrow x \in I \Rightarrow [a, b[\subset I$$

so

$$I = [a, b[.$$

case 3: *If* $a \notin I$ *and* $b \in I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x \leq b \Rightarrow x \in]a, b] \Rightarrow I \subset]a, b]$$

$$\forall x \in \mathbb{R}: x \in]a, b] \Rightarrow a < x \leq b \Rightarrow x - a > 0$$

By putting $\delta = x - a$ *in (2) we get* $x > a'$ *and since* $a, a' \in I$, *then:*

$$a' < x \leq b \Rightarrow x \in I \Rightarrow]a, b] \subset I$$

So

$$I =]a, b]$$

case 4: *If* $a \notin I$ *and* $b \notin I$*, Then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x < b \Rightarrow x \in]a, b[\Rightarrow I \subset]a, b[$$

$$\forall x \in \mathbb{R}: x \in]a, b[\Rightarrow a < x < b \Rightarrow x - a > 0 \text{ and } b - x > 0.$$

By putting $\varepsilon = b - x$ *in (1) and* $\delta = x - a$ *in (2) we get* $x < b'$ *and* $a' < x$*, since* $a', b' \in I$*, then:*

$$a' < x \leq b' \Rightarrow x \in I \Rightarrow]a, b[\subset I.$$

So

$$I =]a, b[.$$

In the same way we prove that I is a interval in the other cases.

Chapter two: Complex numbers

2.1 Definitions and properties

Definition 2.1

We call a complex number and denote it z, for each ordered pair (x, y) of real numbers.

$\cdot$ the x component is called the real part of z and we denote it as $\operatorname{Re}(z)$

$\cdot$ the y component is called the imaginary part of z and we denote it as $\operatorname{Im}(z)$

$$\cdot\ (x, y) = \left(x', y'\right) \Longleftrightarrow \begin{cases} x = x' \\ \text{and} \\ y = y' \end{cases}$$

$\cdot$ the set of complex number we denote $\mathbb{C}$ and provided with two opirations:

1) Addition $(+)$: $(x, y) + \left(x', y'\right) = \left(x + x', y + y'\right)$ where $0_{\mathbb{C}} = (0, 0)$.

2) Multiplication $(\times)$: $(x, y) \times \left(x', y'\right) = \left(xx' - yy', xy' + yx'\right)$ where $1_{\mathbb{C}} = (1, 0)$.

Remarks

1) the set of complex numbers is a commutative field.

2) the neutral element is $0_{\mathbb{C}} = (0, 0)$ we denote it as 0 and the unit element is $1_{\mathbb{C}} = (1, 0)$ we denote it as 1.

Notation

1) the complex number $(0, 1)$ is noted i.

Theorem 2.1

1) we have $i^2 = -1$.

2) if $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then $(x, y) = x + iy$

Proof

1) $i^2 = (0, 1) \times (0, 1) = (-1, 0) = -(1, 0) = -1$

2) we have $(x, y) = (x, 0) + (0, 1)(y, 0) = x + iy$.

Definition 2.2

1) the complex number iy with $y \in \mathbb{R}^*$ is called pure imaginary.

2) the conjugate of a complex number $z = x + iy$ ($x \in \mathbb{R}$ and $y \in \mathbb{R}$) is the complex number $\overline{z}$ where $\overline{z} = x - iy$

Properties

z and w are complex numbers, so

1) $\overline{\overline{z}} = z$

2) $z + \overline{z} = 2\operatorname{Re}(z)$

3) $z + \overline{z} = 2\operatorname{Im}(z)$

4) $\overline{z + w} = \overline{z} + \overline{w}$

5) $\overline{z \times w} = \overline{z} \times \overline{w}$

6) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ $(w \neq 0)$

7) $z = \overline{z} \Longleftrightarrow z \in \mathbb{R}$

8) $z = \overline{z} \Longleftrightarrow z$ pure imaginary

Definition 2.3

we call the module of a coplex number z the positive real number $|z|$ where $|z| = \sqrt{z.\overline{z}} = \sqrt{x^2 + y^2}$.

Properties

z and w are complex numbers, so

1) $|\mathrm{Re}(z)| \leq |z|$; $|\mathrm{Im}(z)| \leq |z|$ 3) $|z.w| = |z|\,|w|$

2) $|z| = 0 \iff z = 0$ 4) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ $(w \neq 0)$

5) $|z+w| \leq |z| + |w|$ (Triangle inequality)

2.2 The trigonometric form

Definition 2.4

Let $z \in \mathbb{C}^*$; There exists a class of real $\theta + 2\pi k\,(k \in \mathbb{Z})$ (or $\theta\,[2\pi]$) where

$$z = |z|\,e^{i\theta}$$

we notice $Arg\,(z) = \theta\,[2\pi]$; $|z| = \rho$ $(\rho > 0)$,then:

$$z = \underbrace{x+iy}_{\text{Algebraic form}} = \underbrace{\rho e^{i\theta}}_{\text{exponentiel form}} = \underbrace{\rho\,(\cos\theta + \sin\theta)}_{\text{Trigonometric form}}$$

where

$$\rho = \sqrt{x^2+y^2}\ ;\ \cos\theta = \frac{x}{\rho}\ ;\ \sin\theta = \frac{y}{\rho}.$$

Remark Let $z \in \mathbb{C}^*$ where $z = x+iy$, so

$$Arg(z) = \theta = \begin{cases} \frac{\pi}{2} & if\ \ x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & if\ \ x = 0 \text{ and } y < 0 \\ \arctan\frac{y}{x} & if\ \ x > 0 \\ \arctan\frac{y}{x} + \pi & if\ \ x < 0 \end{cases}.$$

Properties

Let $z = \rho e^{i\theta}$ and $w = re^{i\varphi}$

1) $z.w = \rho r e^{i(\theta+\varphi)}$ 3) $\frac{1}{z} = \frac{1}{\rho}e^{-i\theta}$

2) $\frac{z}{w} = \frac{\rho}{r}e^{i(\theta-\varphi)}$ 4) $\overline{z} = \rho e^{-i\theta}$

5) $z^n = \rho^n e^{i\theta n} = \rho^n\,(\cos n\theta + \sin n\theta)$ (De moivre's formula ($Abraham\ De\ Moivre(1667-1754)$))

6) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$; $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ (Euler formula)

2.3 Application of complex numbers to trigonometry

2.3.1 Calculates $\cos nx$ and $\sin nx$ based en $\cos x$ and $\sin x$

we have:

$$\begin{aligned} \cos nx + i\sin nx &= (\cos x + i\sin x)^n \\ &= \sum_{k=0}^{n} C_n^k\,(i)^k \cos^{n-k} x \sin^k x \\ &= C_n^0 \cos^n x - C_n^2 \cos^{n-2} x \sin^2 x + C_n^4 \cos^{n-4} x \sin^4 x + ... \\ &\quad + i\left(C_n^1 \cos^{n-1} x \sin x - C_n^3 \cos^{n-3} x \sin^3 x + C_n^5 \cos^{n-5} x \sin^5 x + ...\right). \end{aligned}$$

So

$$\begin{cases} \cos nx = C_n^0 \cos^n x - C_n^2 \cos^{n-2} x \sin^2 x + C_n^4 \cos^{n-4} x \sin^4 x - C_n^6 \cos^{n-6} x \sin^6 x + ... \\ \sin nx = C_n^1 \cos^{n-1} x \sin x - C_n^3 \cos^{n-3} x \sin^3 x + C_n^5 \cos^{n-5} x \sin^5 x + ... \end{cases}$$

or

$$\cos nx = \sum_{p=0}^{\left[\frac{n}{2}\right]} (-1)^p \, C_n^{2p} \cos^{n-2p} x \sin^{2p} x$$

$$\sin nx = \sum_{p=1}^{\left[\frac{n+1}{2}\right]} (-1)^{p-1} \, C_n^{2p-1} \cos^{n-2p+1} x \sin^{2p-1} x$$

where $[\frac{n}{2}]$ denotes the integer part of the rational number $\frac{n}{2}$.

2.3.2 Linearization of trigonometric polynomials

For obtain linearization of $\cos^n x$ and $\sin^n x$ we use:

$$\forall k \in \mathbb{Z}; \forall x \in \mathbb{R} : \cos kx = \frac{e^{ikx} + e^{-ikx}}{2}; \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i} \text{ (Euler formula)}$$

Example: write in linear form $\cos^3 x, \sin^4 x$

we have

$$\begin{aligned}
\cos^3 x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \\
&= \frac{1}{8}\left(2\left(e^{3ix} + e^{-3ix}\right) + 3\left(e^{ix} + e^{-ix}\right)\right) \\
&= \frac{1}{8}\left(2\cos 3x + 3\left(2\cos x\right)\right) \\
&= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.
\end{aligned}$$

$$\begin{aligned}
\sin^4 x &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^4 \\
&= \frac{1}{16}\left(2\left(e^{4ix} + e^{-4ix}\right) + 4 \times 2\left(e2^{ix} + e^{-2ix}\right) + 6\right) \\
&= \frac{1}{16}\left(2\left(e^{4ix} + e^{-4ix}\right) - 4 \times 2\left(e2^{ix} + e^{-2ix}\right) + 6\right) \\
&= \frac{1}{16}\left(2\sin 4x - 4\left(2\sin 2x\right) + 6\right) \\
&= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}.
\end{aligned}$$

2.3.3 nth roots of complex number

Definition 2.5 Let $n \in \mathbb{N}^* - \{1\}$

An nth root of complex number a is a complex number z such that $z^n = a$.

Example

We have $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2 = i$, so $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ one of the square roots of the complex number i

Theorem Let $n \in \mathbb{N}^* - \{1\}$

Any nonzero complex number has exactly n distinct nth roots and if $a = re^{i\theta}$, then the solutions to $z^n = a$ are given by $z_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}}$ for $k \in \{0, 1, ..., n-1\}$.

Proof

Suppose that $z = \rho e^{i\alpha}$, so

$$z^n = a \Longleftrightarrow \rho^n e^{i\alpha n} = re^{i\theta}$$

$$\Longleftrightarrow \quad \rho^n = r \quad \text{and} \quad \alpha n = \theta + 2\pi k, k \in \mathbb{Z}.$$
$$\Longleftrightarrow \quad \rho^n = \sqrt[n]{r} \quad \text{and} \quad \alpha = \frac{\theta + 2\pi k}{n}, k \in \mathbb{Z}.$$

The expression for z takes on n different values for $k = 0, 1, ..., n-1$, and the values start to repeat for $k = n, n+1, ...$

Hence the expression for the n nth roots of a:

$$z_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} \text{ for } k \in \{0, 1, ..., n-1\}$$

Remark

The roots lie on a circle of radius $\sqrt[n]{r}$ centred at the origin and spaced out evenly by angles of $\frac{2\pi}{n}$.

Examples

1) The n nth roots of unity are therefore the numbers $z_k = e^{i\frac{\theta+2\pi k}{n}} = \cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}$ for $k \in \{0, 1, ..., n-1\}$

2) Solve in $\mathbb{C}$ the equatuon $z^7 = \overline{z}$.

a) It is clear that 0 is one of the solutions.

b) Suppose that $z \neq 0$ and $z = \rho e^{i\theta}$, so

$$z^7 = \overline{z} \Longleftrightarrow \rho^7 e^{7i\theta} = \rho e^{-i\theta}$$

$$\Longleftrightarrow \begin{cases} \rho^7 = \rho \\ 7\theta = -\theta + 2\pi k \text{ where } k \in \mathbb{Z} \end{cases}$$
$$\Longleftrightarrow \begin{cases} \rho\left(\rho^6 - 1\right) = 0 \\ 8\theta = 2\pi k \text{ where } k \in \mathbb{Z} \end{cases}$$
$$\Longleftrightarrow \begin{cases} \rho = 1 \\ \theta = \frac{\pi k}{4} \text{ where } k \in \{0, 1, 2, 3, 4, 5, 6, 7\} \end{cases}$$

So the set of solutions is $S = \left\{0, e^{i\frac{\pi k}{4}} / k \in \{0, 1, 2, 3, 4, 5, 6, 7\}\right\}$, so

$$S = \left\{0, 1, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, i, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -i, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right\}.$$