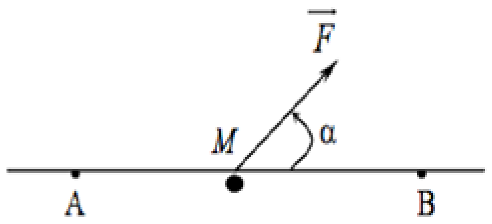
Chapter 3: Energy work in the case of a material point

1/ Work of a force

1.1/ Constant force on a rectilinear movement:

Consider a constant force acting on a material point M. Under the influence of $\vec{\mathbf{F}}$, M moves between points A and B By definition, the work of the force $\vec{\mathbf{F}}$ on the rectilinear displacement AB is given by:

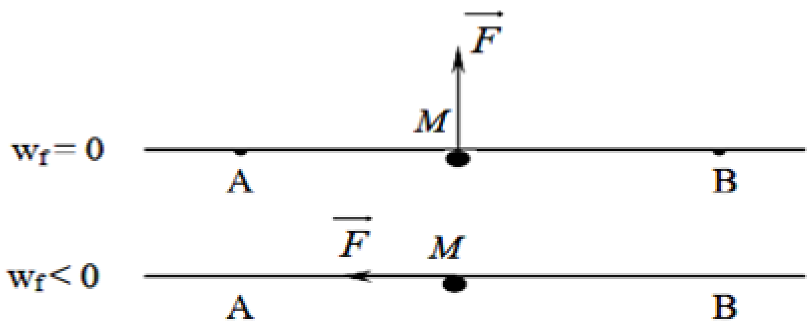


$$W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F.AB\ \cos\alpha$$

α is the angle that makes $\vec{\mathbf{F}}$ with **AB** .

Noticed :

The work is either positive, zero or negative depending on the direction of the force relative to
when moving. The unit of work is the **Joule .**



Example :

Calculate the work necessary to move a mass from point A (x=0m) to point B (x= 10 m) under the effect of the force:

$$F(x) = 5x + 3$$

Corrected:

$$W_A^B = \int_A^B dW = \int_A^B \vec{F}d\vec{l} = \int_0^{10} F(x)dx$$

$$W_A^B = 5\int_0^{10} xdx + 3\int_0^{10} dx$$

The work between A and B is:

$$W_A^B = 5\int_0^{10} x dx + 3\int_0^{10} dx$$

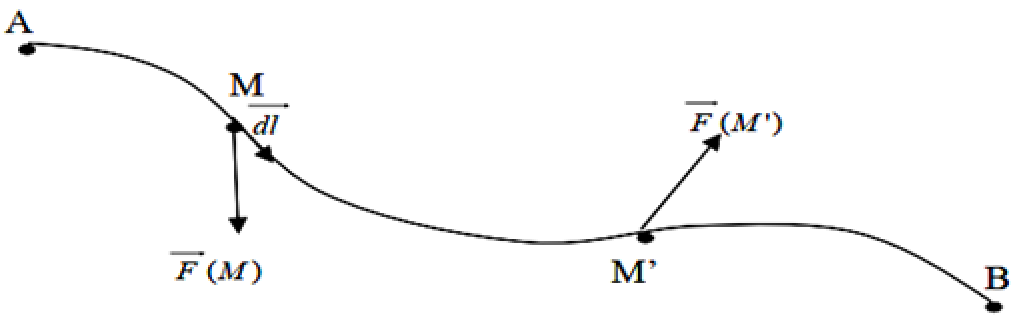
$$W_A^B = 5\left[\frac{x^2}{2}\right]_0^{10} + 3[x]_0^{10}$$

$$W_A^B = 5\left[\frac{10^2}{2} - \frac{0^2}{2}\right] + 3[10-0]$$

$$W_A^B = 250 + 30 = 280\ Joules$$

1.2/ Elementary work:

In case the force $\vec{\mathbf{F}}$ varies during the movement which can be arbitrary, it is no longer possible to use the previous expression. We break down the journey $\overrightarrow{\mathbf{AB}}$ in a succession of elementary movements $\overrightarrow{\mathbf{dl}} = \overrightarrow{\mathbf{MM'}}$ infinitely small and therefore rectilinear.



elementary work

On $\overrightarrow{\mathbf{MM'}}$, strength $\vec{\mathbf{F}}$ can be considered constant; then we define the elementary work given by: $dW_F = \vec{\mathbf{F}}.\vec{dl}$

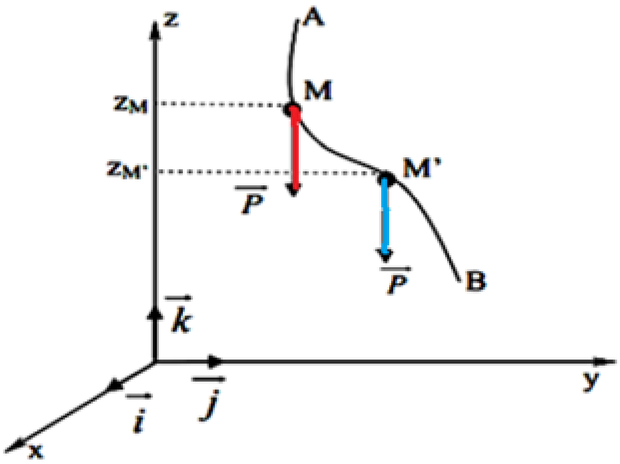
1.3/ Variable force on any displacement:

To obtain the total work on the total displacement, simply add the elementary works:

$$W_{\vec{F}} = \int_A^B \vec{F}\ \vec{dl}$$

1.4/ Work of the force of gravity:

or the following figure, with: $\mathbf{h = Z_M - Z_{M'}}$



Work of the force of gravity

$$w_{\vec{P}} = \int_{M}^{M'} \vec{P}.\vec{dl}$$

$$\vec{P} = -P\vec{K}$$

$$\vec{dl} = dx\vec{i} + dy\,\vec{j} + dz\vec{k} \quad \Rightarrow \vec{P}.\vec{dl} = -PdZ$$

$$\text{donc} \quad w_{\vec{P}} = \int_{M}^{M'} \vec{P}.\vec{dl} = \int_{M}^{M'} -P.dZ = -P(Z_{M'} - Z_M)$$

$$\Rightarrow w_{\vec{P}} = P(Z_{M'} - Z_M) = Ph = mgh$$

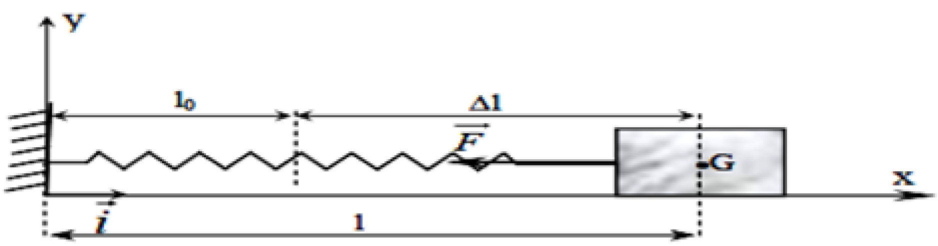
1.5/ Work of an elastic force:

$$\vec{F} = -K\,\Delta l\,\vec{i} = -K(l - l_0)\vec{i} = -kx\,\vec{i}$$

$$dw_{\vec{F}} = \vec{F}.\vec{dl} = -Kx\vec{i}dx\vec{i} = Kxdx = -d(\frac{1}{2}kx^2)$$

When $\vec{F}$ moves from one position x_1 at x_2, we have:

$$w_{\vec{F}} = \int_{x_1}^{x_2} \vec{F}.\vec{dl} = \int_{x_1}^{x_2} -kxdx = -\frac{1}{2}k({x_2}^2 - {x_1}^2)$$



Work of an elastic force

2/ Energy:

2.1/ Kinetic energy:

We define the kinetic energy of a material point M , of mass m and animated with a speed v , by the quantity E_c, such that:

$$E_c = (1/2)\, mv^2$$

Consider a material point M , of mass m , moving between points A and B under the action of an external force $\vec{F}$. According to the fundamental principle of dynamics, we have:

$$\sum \vec{F}_{ext} = m\frac{d\vec{V}}{dt} \Rightarrow \vec{F} = m\frac{d\vec{V}}{dt}$$

The elementary work of $\vec{F}$ is given by:

$$dW_{\vec{F}} = \vec{F}.\vec{dl} = m\frac{d\vec{v}}{dt}\vec{V}\,dt$$

$$\text{car} \quad \vec{V}(t) = \frac{d\vec{l}}{dt} \Rightarrow d\vec{l} = \vec{V}(t).dt$$

$$\text{il vient} \quad dW_{\vec{F}} = \vec{F}.\vec{dl} = d\left(\frac{1}{2}mV^2\right)$$

The work done between points A and B will be:

$$W_{\vec{F}} = \int_A^B \vec{F}.\vec{dl} = \int_A^B d\left(\frac{1}{2}mV^2\right) = \int_A^B dE_C = E_C(B) - E_C(A)$$

2.2/ Kinetic energy theorem:

In a Galilean frame of reference, the variation in kinetic energy of a material point subjected to a set of external forces between a position A and another position B is equal to the sum of the work of these forces between these two points.

$$E_C(B) - E_C(A) = \sum W_{A-B}(\vec{F}_{ext})$$

3/ Conservative and non-conservative forces:

3.1/ Conservative forces:

Forces are said to be conservative when their work does not depend on the path followed but only from the starting point and the arrival point.

Examples : force of gravity, force of weight, spring return force.

The work of these forces can therefore be expressed from a state function called potential energy E_P(function depending only on the state of the system).

*The variation in potential energy between two points A and B is equal to the opposite of the work of the conservative force between these two points.

$$\mathbf{W} = \int_{A}^{B} \vec{F}.\overrightarrow{dl} = E_P(A) - E_P(B) = -\Delta E_P$$

In this case we say that the force derives from a potential energy $\mathbf{E_P}$is:

$$\vec{F} = -\overrightarrow{grad}E_p$$

$$\vec{F} = -\left(\frac{\partial E_P}{\partial x}\vec{i} + \frac{\partial E_P}{\partial y}\vec{j} + \frac{\partial E_P}{\partial z}\vec{k}\right)$$

All the forces we have seen, weight, gravitational force and elastic are forces that derive from a potential.

3.2/ Non-conservative forces:

These are all the other forces whose work depends on the path followed. They do not derive from potential energy.

Friction forces can be cited as an example. The work of these forces is always resistant (negative work). Let us take the case of a solid type friction force . $\vec{F}$This force continually opposes the movement and its norm $\mathbf{F}$is constant.

The work of the solid friction force gives:

$$\mathbf{W} = \int_{A}^{B} \vec{F}.\overrightarrow{dl} = -F\int_{A}^{B} dl = -F.L_{AB}$$

The length $\mathbf{L_{AB}}$is the distance actually traveled between A and B. This distance obviously depends on the path followed.

3.3/ Mechanical energy:

The mechanical energy of a system is equal to the sum of the kinetic and potential energies:

$$E_m = E_C + E_P$$

Mechanical energy theorem

- If a system is not subjected to any non-conservative force (no friction force), the mechanical energy is conserved:

$$\Delta E_m = 0 \quad \Leftrightarrow \quad \text{forces conservatives}$$

- If the system is subjected to non-conservative forces (friction forces) the mechanical energy is dissipated, that is to say the variation in mechanical energy of a system between two points A and B is equal to the sum of the work of the non-conservative forces applied to the system between these two points

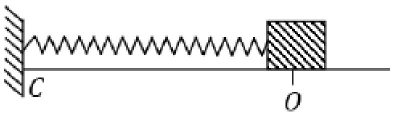
$$\Delta \mathbf{E_m} = \mathbf{E_m}(\mathbf{B}) - \mathbf{E_m}(\mathbf{A}) = \sum \mathbf{W}(\textbf{forces non conservatives})$$

Application exercises:

Exercise 1:

A mass $m$is linked to a spring of stiffness $\boldsymbol{k}$, the other end of the spring is linked to point c. the mass m can slide on the horizontal surface. First of all, the mass is at rest at the equilibrium point **O**.

1. We assume that there is no friction, we move the mass $m$from the point $O$to point $\boldsymbol{A}$, such that $\boldsymbol{OA} = \boldsymbol{a}$, determine the work of the return force of the spring, as m it moves from $\boldsymbol{O}$to A. Then determine the speed of $\boldsymbol{m}$at the point $\boldsymbol{O}$.
2. Same questions as 1, but now we assume that there is friction, we give the dynamic coefficient $\boldsymbol{\mu_c}$.

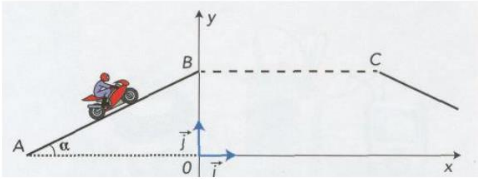


Exercise 2:

On March 31, 2008, Australian Robbie Maddison broke his own motorcycle long jump record.
Consider a springboard inclined at an angle $\alpha = 27°$to the horizontal. We consider Maddison traveled the AB springboard with a speed of constant value equal to 160 km/h/. At point B, he took off for a one-reach jump$BC = 107\ m$.
Between B and C, any force other than weight is assumed to be negligible.
We choose the altitude of point A as a reference for the potential energies of gravity.
1. Express the mechanical energy of the system {biker + motorcycle} as a function of the value of the speed V and the altitude y.
2. Calculate the kinetic energy of the system at point A.
3.a. Express the altitude $y_B$of point B in terms of AB and α.
b. Deduce the expression for the variation in gravitational potential energy of the system, when the system passes from point A to point B. Calculate this energy variation.
vs. How does the mechanical energy of the system change when it goes from A to B? Justify the answer.
4. How does the mechanical energy of the system change when it goes from B to C? Justify the answer.

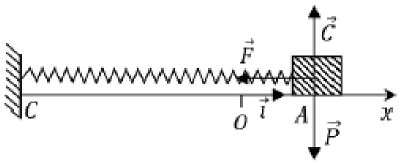
5. Deduce its speed at point C.

Data: • intensity of gravity: $g = 9.81\ N.kg^{-1}$, mass of the system: $m = 180kg$, $AB = 7.86m$.



Solution exercise 1:

CORRIGÉ :



1) $W_{\vec F} = \int \vec F . d\vec r = \int_a^0 (-kx\,\vec i).\left(dx\,\vec i\right) = \frac{ka^2}{2}$

$\Delta E_c = W_{\vec F} + W_{\vec C} + W_{\vec P}$

$$W_{\vec C} = W_{\vec P} = 0 \Longrightarrow \frac{1}{2} m\, v_O^2 - \frac{1}{2} m\, v_A^2 = \frac{ka^2}{2} \Longrightarrow v_O = a\sqrt{\frac{k}{m}}$$

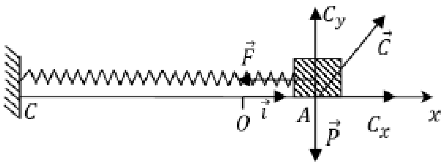
2) $W = W_{\vec F} + W_{\vec C}$

$W_{\vec C} = \int \vec C . d\vec r = \int_a^0 (C_x\,\vec i + C_y\,\vec j).\left(dx\,\vec i\right) = -a\,C_x$

$\Rightarrow W = \frac{ka^2}{2} - a\,C_x$ avec $C_x = \mu_c C_y = \mu_c mg$

Soit $W = \frac{ka^2}{2} - a\mu_c mg$ et $\Delta E_c = \sum W$

$\Rightarrow \frac{1}{2} m v_0^2 = \frac{ka^2}{2} - a\mu_c mg = a\sqrt{\frac{k}{m} - \frac{2\mu_c g}{a}}$



Solution exercise 2:

1. L'énergie mécanique vaut : $Em = Ec + Epp = \frac{1}{2} mV^2 + m.g.y$

2. $Ec = \frac{1}{2} mV^2 = \frac{180}{2}\left(\frac{160 \times 1000}{3600}\right) = 177777\ J$

3. a) $y_B = AB.\sin\alpha$

   b) $\Delta Epp = Epp\ finale - Epp\ initiale = m.g.y - 0 = 180 \times 9{,}81 \times 7{,}86 \sin 27 = 6301\ J$

   c) Si la vitesse reste constante l'énergie cinétique reste inchangée.
   L'énergie potentielle passe de 0 à 6301 J quand le motard va de A à B.
   Donc l'énergie mécanique augmente : En B elle vaut :
   $Em_B = Ec + Epp = 177777 + 6301 = 184078\ J$

4. On a $y_B = y_C$ et de ce fait l'énergie potentielle est la même.
   Si l'on néglige les forces de frottement de l'air, la vitesse en C sera identique à la vitesse en B. L'énergie cinétique en C sera la même qu'en B. Donc l'énergie mécanique en C sera la même qu'en B. $Em_C = Em_B = 184078\ J$

5. $V_C = V_B = 160\ km/h$