University of Oum-El-Bouaghi
Faculty of SENV, department of M.I.
Final exam in Algebra 4
May 2026

Exercise 1 (a: 6, b: 4) Let $E = \mathbb{R}_2[X]$ and $\varphi : E \times E \to \mathbb{R}$, such that $\varphi(f, g) = \sum_{i=0}^{2} f(i)\, g(i)$.

a) Prove that φ is a scalar product.

b) Determine the associated matrix in the canonical basis.

Exercise 2. **(4 points)** Let q be a quadratic form on a vector space E. Give an example where $\ker q \subsetneqq I$, the set of isotropic vectors.

Exercise 3. **(a:3,b:3)** Let $\varphi_{1,}\varphi_2, \varphi_3 \in (\mathbb{R}^3)^*$, such that

$$\varphi_{1,}(x, y, z) = x + y + z,\ \varphi_2(x, y, z) = x - y + z,\ \varphi_{3,}(x, y, z) = x - y - z$$

a) Prove that $\{\varphi_{1,}\varphi_2, \varphi_3\}$ is a basis for $(\mathbb{R}^3)^*$.

b) Determine the antedual basis.

Solution

Exercise 1.(a: 6, b: 4) Let $E = \mathbb{R}_2[X]$ and $\varphi : E \times E \to \mathbb{R}$, such that $\varphi(f, g) = \sum_{i=0}^{2} f(i) g(i)$.

a) To prove that φ is a scalar product, we prove that it is a positive definite symmetric bilinear form:

i) **(1 point)** Since $f(X), g(X) \in \mathbb{R}_2[X]$, then $f(i), g(i) \in \mathbb{R}$ for $i = 0, 1, 2$. Therefore $\varphi(f, g) = \sum_{i=0}^{2} f(i) g(i) \in \mathbb{R}$.

ii) **(1 point)** Since $f(i), g(i) \in \mathbb{R}$ for $i = 0, 1, 2$, then $f(i) g(i) = g(i) f(i)$ for $i = 0, 1, 2$, which yields to

$$\varphi(f, g) = \sum_{i=0}^{2} f(i) g(i) = \sum_{i=0}^{2} g(i) f(i) = \varphi(g, f)$$

iii) **(1 point)** Since we prove that φ is symmetric, then we need to prove linearity for only one side.

iii) 1) For every $f(X), g(X), h(X) \in \mathbb{R}_2[X]$, we have

$$\begin{aligned}
\varphi(f + h, g) &= \sum_{i=0}^{2} (f + h)(i) g(i) = \sum_{i=0}^{2} (f(i) + h(i)) g(i) \\
&= \sum_{i=0}^{2} f(i) g(i) + h(i) g(i) = \sum_{i=0}^{2} f(i) g(i) + \sum_{i=0}^{2} h(i) g(i) \\
&= \varphi(f, g) + \varphi(h, g)
\end{aligned}$$

iii) 2) For every $f(X), g(X) \in \mathbb{R}_2[X]$ and $\alpha \in \mathbb{R}$, we have

$$\varphi(\alpha f, g) = \sum_{i=0}^{2} (\alpha f)(i) g(i) = \sum_{i=0}^{2} \alpha f(i) g(i) = \alpha \sum_{i=0}^{2} f(i) g(i)$$

iv) **(1 point)** For every $f(X) \in \mathbb{R}_2[X]$, the form is positive because it is the sum of squares:

$$\varphi(f,f) = \sum_{i=0}^{2} (f(i))^2 \geq 0$$

v) **(2 points)** 1Let $f(X) = aX^2 + bX + c$, then,

$$f(0) = c,\ f(1) = a+b+c,\ f(2) = 4a+2b+c,$$

which yields to

$$\varphi(f,f) = c^2 + (a+b+c)^2 + (4a+2b+c)^2 = 0 \Rightarrow \begin{cases} c &= 0 \\ a+b &= 0 \\ 4a+2b &= 0 \end{cases} \Rightarrow a = b = 0$$

which means φ is definite, i.e.

$$\varphi(f,f) = 0 \Rightarrow f = 0$$

b) Let $M_\varphi = (a_{ij})_{3\times 3}$ be the associated matrix of φ in the canonical basis $\{1, X, X^2\}$

$$\begin{aligned}
f(X) &= g(X) = 1 = f(0) = f(1) = f(2) \Rightarrow a_{11} = \varphi(1,1) = 1\times 1 + 1\times 1 + 1\times 1 = 3 \\
f(X) &= 1,\ g(X) = X \Rightarrow a_{21} = a_{12} = \varphi(1,X) = 1\times 0 + 1\times 1 + 1\times 2 = 3 \\
f(X) &= 1,\ g(X) = X^2 \Rightarrow a_{31} = a_{13} = \varphi\left(1,X^2\right) = 1\times 0 + 1\times 1 + 1\times 4 = 5 \\
f(X) &= X = g(X) \Rightarrow a_{22} = \varphi(X,X) = 0\times 0 + 1\times 1 + 2\times 2 = 5 \\
f(X) &= X,\ g(X) = X^2 \Rightarrow a_{32} = a_{23} = \varphi\left(X,X^2\right) = 0\times 0 + 1\times 1 + 2\times 4 = 9 \\
f(X) &= X^2 = g(X) \Rightarrow a_{33} = \varphi\left(X^2,X^2\right) = 0\times 0 + 1\times 1 + 4\times 4 = 17
\end{aligned}$$

Therefore, the associated matrix

$$M_\varphi = \begin{pmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix}$$

Exercise 2. **(4 points)** Let q be a quadratic form on a vector space E. If we take $E = \mathbb{R}^3$ and $q(x, y, z) = x^2 + y^2 - z^2$, then we have

$$\ker q = \{0\} \underset{\neq}{\subseteq} I = \left\{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 = z^2\right\},$$

it is easy to see that in addition to $0 = (0, 0, 0)$, all Pythagorean triples (x, y, z) satisfy the relation $q(x, y, z) = 0$

Exercise 3. **(a:3,b:3)** Let $\varphi_{1,}\varphi_2, \varphi_3 \in (\mathbb{R}^3)^*$, such that

$$\varphi_{1,}(x, y, z) = x + y + z,\ \varphi_2(x, y, z) = x - y + z,\ \varphi_{3,}(x, y, z) = x - y - z$$

a) Since the set contains 3 elements, then it is sufficient to prove that $\left\{\varphi_{1,}\varphi_2, \varphi_3\right\}$ is free. We can prove it by the definition of linear independence or to consider φ_1, φ_2, φ_3 as vectors $(1, 1, 1)$, $(1, -1, 1)$, $(1, -1, -1)$, then calculating the determinant

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = 4$$

b) TO find the antedual basis, we calculate the inverse matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1}$,

then we take its columns with respect of order:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$v_1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), v_2 = \left(0, -\frac{1}{2}, \frac{1}{2}\right), v_3 = \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$$