University of Oum-El-Bouaghi
Faculty of SENV, department of M.I.
Final exam in Algebra 4 (for debtors)
Mai 2026

Instructions: Any method not mentioned in the questions will not be considered. For example, if we say "deduce", then any method of computation not related to the previous question will be rejected.

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1. **Exercise.**(a: 2, b: 2) In a Euclidean space, prove the following propositions:

   (a) The eigenvectors associated with distinct eigenvalues are pairwise orthogonal.

   (b) Orthogonal vectors are free.

2 **Exercise**. (a: 6, b: 3, c: 3)

   (a) In an orthonormal basis of eigenvectors diagonalize the following quadratic form:

$$q_a(X) = 3x_1^2 - 2x_1x_2 + 4x_2^2 + 4ax_3^2 \quad / \; X = (x_1, x_2, x_3)$$

   (b) Based on the values of the real number a, deduce the signature and the rank of q_a. Is the form positive, negative, or indefinite?

   (c) Based on the values of the real number a, determine the isotropic vectors and the kernel of q_a.

3 **Exercise.**(a: 2, b: 2) Let

$$\varphi(f,g) = \int_0^1 f(x)\,dx \times \int_0^1 g(x)\,dx \quad \text{for all } (f,g) \in E^2 = (\mathbb{R}_1[x])^2$$

   (a) Prove that φ is a symmetric bilinear form on E

   (b) Determine the associated matrix in the canonical basis of E.

Solution

Exercise 1.(a: 2, b: 2)

a) The eigenvectors associated with distinct eigenvalues are pairwise orthogonal. Let $\langle \cdot, \cdot \rangle$ denotes the scalar product over the Euclidean space E.

Let v_1 and v_2 be two eigenvectors of a symmetric matrix A associated with distinct eigenvalues λ_1 and λ_2, and let us suppose that v_1 and v_2 are not orthogonal, that means that $\langle v_1, v_2 \rangle \neq 0$. Then from the identity $\langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle$, we get

$$\langle \lambda_1 v_1, v_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle$$

Therefore, we get

$$\lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

Since $\langle v_1, v_2 \rangle \neq 0$, that gives $\lambda_1 = \lambda_2$ which is absurd.

b) Let v_1 and v_2 be two nonzero orthogonal vectors, such that there exists a scalar α with $v_2 = \alpha v_1$, then

$$0 = \langle v_1, v_2 \rangle = \langle v_1, \alpha v_1 \rangle = \alpha \langle v_1, v_1 \rangle$$

Since $\langle v_1, v_1 \rangle \neq 0$, then $\alpha = 0$, which means that v_1 and v_2 are free.

Exercise 2. (a: 6, b: 3, c: 3)

$$q_a(X) = 3x_1^2 - 2x_1x_2 + 4x_2^2 + 4ax_3^2 \;\; / \; X = (x_1, x_2, x_3)$$

a) Then the associated matrix is

$$M_{q_a} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 4a \end{pmatrix}$$

By calculation of characteristic polynomial, you find the eigenvectors associated with their eigenvalues **(1,5+1,5):**

$$v_1 = \begin{pmatrix} \frac{1}{2}\sqrt{5} + \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \leftrightarrow \frac{7}{2} - \frac{1}{2}\sqrt{5}, v_2 = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}\sqrt{5} \\ 1 \\ 0 \end{pmatrix} \leftrightarrow \frac{7}{2} + \frac{1}{2}\sqrt{5}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \leftrightarrow 4a$$

Since all the eigenvalues are distinct, then according to exercise 1), a), the eigenvectors are pairwise orthogonal. Then, we just need to normalize them by the following:

$$\begin{aligned}
\epsilon_1 &= \frac{1}{\|v_1\|}v_1 = \frac{5+\sqrt{5}}{2}\begin{pmatrix} \frac{1}{2}\sqrt{5}+\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5+3\sqrt{5}}{2} \\ \frac{5+\sqrt{5}}{2} \\ 0 \end{pmatrix} \\
\epsilon_2 &= \frac{1}{\|v_2\|}v_2 = \frac{5-\sqrt{5}}{2}\begin{pmatrix} \frac{1}{2}-\frac{1}{2}\sqrt{5} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5-3\sqrt{5}}{2} \\ \frac{5-\sqrt{5}}{2} \\ 0 \end{pmatrix} \\
\epsilon_3 &= \frac{1}{\|v3\|}v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

(1,5)

In the orthonormal basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$, we get:

$$P^{-1}M_{q_a}P = P^tM_{q_a}P = D = \begin{pmatrix} \frac{7-\sqrt{5}}{2} & & \\ & \frac{7+\sqrt{5}}{2} & \\ & & 4a \end{pmatrix} \quad \textbf{(0,5)}$$

which gives

$$q_a(X) = \left(\frac{7-\sqrt{5}}{2}\right)y_1^2 + \left(\frac{7+\sqrt{5}}{2}\right)y_2^2 + 4ay_3^2 \quad \textbf{(0,5)}$$

where

$$X = y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3 \quad \textbf{(0,5)}$$

b) Based on the values of the real number a, the signatures of q_a are:

$$s(q_a) = (P, N) = \begin{cases} (3,0) & \text{for} \quad a > 0 \Rightarrow q_a \text{ is positive definite} \\ (2,0) & \text{for} \quad a = 0 \Rightarrow q_a \text{ is positive skew definite} \\ (2,1) & \text{for} \quad a < 0 \Rightarrow q_a \text{ is indefinite} \end{cases} \quad \textbf{(2)}$$

From the previous study of the signature, the ranks of q_a

$$r(q_a) = \begin{cases} 3 & \text{for} \quad a > 0 \text{ or } a < 0 \\ 2 & \text{for} \quad a = 0 \end{cases} \quad \textbf{(1)}$$

c) Based on the values of the real number a, the set I of isotropic vectors of q_a

$$\begin{array}{rll} I = \{0\} & \text{for} & a > 0 \\ I = \{\alpha\epsilon_3 / \alpha \in \mathbb{R}\} & \text{for} & a = 0 \\ I = \left\{y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3 / \left(\frac{7-\sqrt{5}}{2}\right) y_1^2 + \left(\frac{7+\sqrt{5}}{2}\right) y_2^2 = -4ay_3^2\right\} & \text{for} & a < 0 \end{array} \quad \textbf{(1,5)}$$

The kernel of q_a. **(1,5)**

$$\begin{aligned} \ker q_a &= \{0\} \text{ for } \quad a > 0 \text{ or } a < 0 \\ \dim \ker q_a &= 1 \text{ for } \quad a = 0 \end{aligned}$$

Exercise 3.(a: 2, b: 2) We have $l_1 = \int_0^1 f(x)\,dx$, $l_2 = \int_0^1 f(x)\,dx$ are two linear forms on a vector space E over a field $\mathbb{K}$, then the product is a form **(0,5)**

$$\varphi(f,g) = \int_0^1 f(x)\,dx \times \int_0^1 g(x)\,dx \text{ for all } (f,g) \in E^2 = (\mathbb{R}_1[x])^2$$

a) We prove that φ is a symmetric bilinear form on E. First let us prove that the form is symmetric. **(0,5)**

1. (a) Since $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ are scalars, then for all $(f,g) \in E^2 = (\mathbb{R}_1[x])^2$, we have

$$\varphi(f,g) = \int_0^1 f(x)\,dx \times \int_0^1 g(x)\,dx = \varphi(f,g) = \int_0^1 g(x)\,dx \times \int_0^1 f(x)\,dx = \varphi(g,f) \;.$$

Since we prove that the form is symmetric, then we need to prove linearity for only one side: **(1)**

$$\begin{aligned} \varphi(f+h,g) &= \int_0^1 (f(x)+h(x))\,dx \times \int_0^1 g(x)\,dx \text{ for all } f,g,h \in E = \mathbb{R}_1[x] \\ &= \left(\int_0^1 f(x)\,dx + \int_0^1 h(x)\,dx\right) \times \int_0^1 g(x)\,dx \\ &= \int_0^1 f(x)\,dx \times \int_0^1 g(x)\,dx + \int_0^1 h(x)\,dx \times \int_0^1 g(x)\,dx \\ &\quad \varphi(f,g) + \varphi(h,g) \end{aligned}$$

$$\begin{aligned}\varphi(\alpha f, g) &= \int_0^1 (\alpha f(x))\,dx \times \int_0^1 g(x)\,dx \ \text{ for all } (f,g) \in E^2 = (\mathbb{R}_1[x])^2, \alpha \in \mathbb{R} \\ &= \int_0^1 \alpha f(x)\,dx \times \int_0^1 g(x)\,dx = \alpha \int_0^1 f(x)\,dx \times \int_0^1 g(x)\,dx = \alpha\varphi(f,g)\end{aligned}$$

(b) The associated matrix in the canonical basis of E. Let $A = (a_{ij})_{2\times 2}$. Then,

$$\begin{aligned}a_{11} &= \varphi(1,1) = \int_0^1 1dx \times \int_0^1 1dx = 1 \\ a_{21} &= a_{12} = \int_0^1 1dx \times \int_0^1 xdx = \frac{1}{2} \\ a_{22} &= \int_0^1 xdx \times \int_0^1 xdx = \frac{1}{4}\end{aligned}$$

Then

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$