Chapter 2: Bilinear forms and quadratic forms, orthogonality and isotropy

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1 Introduction

The notion of bilinear form is defined on vector spaces, which are special cases of bilinear applications on a Cartesian product of two vector spaces in a vector space (where all the spaces involved are defined on the same body). These forms are closely linked to linear applications. The knowledge associated with the latter makes it possible to shed light on the structure of a bilinear form. Some bilinear forms are also scalar products. Scalar products (on finite or infinite dimensional vector spaces) are widely used, in all branches of mathematics, to define a distance.

Classical, relativistic or quantum physics uses this formal framework. Geometry uses the scalar product to define distance, orthogonality, angle, ... Number theory uses quadratic forms to demonstrate or solve certain purely algebraic problems. Sometimes, linking mathematical branches, such as number theory and algebraic geometry, such as the search for solutions to a Diophantine equation. Some of them are written as the search for the roots of a polynomial equation with several variables and integer coefficients. The solutions sought are those that are expressed only with integers. A famous and difficult example is Fermat's great theorem. The equation is written $x^n + y^n = z^n$ (for $n = 2$, the solutions are the Pythagorean triplets, which are called Fermat's two-square theorem). The solutions can be seen as points of intersection between $\mathbb{Z}^3$ and a surface of a geometric space of dimension three. To be compatible with the ministerial program, we limit ourselves to bilinear forms on a finite-dimensional vector space (i.e. the Cartesian product of a vector space in itself), in particular, the quadratic forms taken are those of the symmetric bilinear forms.

2 Bilinear forms

Definition 1 *a bilinear form φ is a map defined on the cartesian product $E \times E \longrightarrow \mathbb{K}$ and satisfies the following conditions*

1. $\forall x,\ y,\ x',\ y' \in E$,

$$\begin{aligned}\varphi(x+x',y) &= \varphi(x,y)+\varphi(x',y)\\ &\quad \textit{and}\\ \varphi(x,y+y') &= \varphi(x,y)+\varphi(x,y').\end{aligned}$$

2. $\forall x,\ y \in E,\ \forall \lambda \in \mathbb{K}$,

$$\varphi(\lambda x,y) = \lambda\varphi(x,y) = \varphi(x,\lambda y).$$

In addition, if $\forall x,\ y \in E,\ \varphi(x,y) = \varphi(y,x)$*, then, the form is said to be symmetric. The form is said to be alternate (anti-symmetric), if*

$$\forall x,\ y \in E,\ \varphi(x,y) = -\varphi(y,x),$$

Note that if the form is symmetric, then, it is sufficient to verify the linearity only on one side.

Example 2 *Determine the bilinear forms and the symmetric ones among the following maps*

1. $\varphi : \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K},\ \varphi(x,y) = xy.$
2. $\varphi : \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K},\ \varphi(x,y) = x+y.$
3. $\varphi : \mathbb{K}^2\times\mathbb{K}^2 \longrightarrow \mathbb{K},\ \forall x = (x_1,x_2),\ y = (y_1,y_2) \in \mathbb{K}^2$,

$$\varphi(x,y) = x_1y_2 + x_2y_2.$$

4. $\varphi : \mathbb{K}^2\times\mathbb{K}^2 \longrightarrow \mathbb{K},\ \forall x = (x_1,x_2),\ y = (y_1,y_2) \in \mathbb{K}^2$,

$$\varphi(x,y) = x_1y_2 + x_2y_1.$$

5. $\varphi : \mathbb{K}^2\times\mathbb{K}^2 \longrightarrow \mathbb{K},\ \forall x = (x_1,x_2),\ y = (y_1,y_2) \in \mathbb{K}^2$,

$$\varphi(x,y) = x_1 + y_2 + x_2y_2.$$

Answer: *Applying the previous definition, we have the first and the fourth are symmetric bilinear forms while the third is a non-symmetric bilinear form, on the other hand, the second and the fifth are not bilinear.*

From the previous example, we can ask ourselves if there is an easier way to know bilinear forms from its appearance? The answer is yes. In the following we will look for an algebraic expression for a bilinear form, so, it is enough to compare a given form with this expression in the same basis.

2.1 The algebraic expression of a bilinear form and the associated matrix

Let φ be a bilinear form on a vector space E with a basis $\{v_1, \cdots, v_n\}$. Then, for $i,\, j = \overline{1, n}$, we have $\varphi(v_i, v_j) \in \mathbb{K}$. Thus there exists a matrix $(\varphi(v_i, v_j))_{n\times n} \in M(n, \mathbb{K})$ is called the matrix associated with φ in the indicated basis. On the other hand, $\forall x,\, y \in E$, $\exists x_1, \ldots, x_n,\, y_1, \ldots, y_n \in \mathbb{K}$, such that

$$x = x_1 v_1 + \cdots + x_n v_n,\; y = y_1 v_1 + \cdots + y_n v_n$$

Applying linearity several times to both sides, we obtain the following double sum

$$\varphi(x, y) = \sum_{i,j=1}^{n} \varphi(v_i, v_j)\, x_i y_j \tag{1}$$

The expression 1 is called the algebraic expression of φ, or more often called the coordinate expression. Thus the terms of the sum for which we define φ contain only mixed products $x_i y_j$ with coefficients in $\mathbb{K}$. As we can write the expression 1 in matrix form:

$$\varphi(x, y) = x^\top A y, \text{ où } A = (\varphi(v_i, v_j))_{n\times n}.$$

The matrix$A = (\varphi(v_i, v_j))_{n\times n}$ is called *Gram matrix.*

Let's put $\varphi(v_i, v_j) = a_{ij}$ for $i,\, j = \overline{1, n}$. If we change the base of E to the basis $\{u_1, \cdots, u_n\}$, so for all $k,\, l = \overline{1, n}$, we obtain

$$u_k = p_{1k} v_1 + \cdots + p_{nk} v_n,\; u_l = p_{1l} v_1 + \cdots + p_{nl} v_n$$

Therefore, in the same previous manner, we obtain

$$\varphi(u_k, u_l) = \sum_{i,j=1}^{n} a_{ij} p_{ik} p_{jl} = \sum_{i,j=1}^{n} p'_{ki} a_{ij} p_{jl}, \text{ où } p'_{ki} = p_{ik}$$

Thus, the matrix $B = (\varphi(u_k, u_l))_{n\times n}$ the matrix associated with the bilinear form φ in the new basis is given by

$$B = P^T A P.$$

Example 3 *Give the matrix associated with the bilinear form φ defined on $\mathbb{R}^3$ by $\varphi(x, y) = x_1 y_2 + x_2 y_3 + x_3 y_3$.*

Let $\{v_1 = (1, 1, -1), v_2 = (1, -1, 0), v_3 = (0, 1, 1)\}$ be basis for $\mathbb{R}^3$. Calculate the matrix associated with φ in this basis by two different methods.

Answer:

1. *Direct method: We put $b_{ij} = \varphi(v_i, v_j)$ for $i,\, j = 1, 2, 3$. Then, we get*

$$\begin{aligned}
b_{11} &= \varphi(v_1, v_1) = 1 \times 1 + 1 \times (-1) + (-1) \times \times (-1) = 1 \\
b_{12} &= \varphi(v_1, v_2) = 1 \times (-1) + 1 \times 0 + (-1) \times 0 = -1 \\
b_{13} &= \varphi(v_1, v_2) = 1 \times 1 + 1 \times 1 + (-1) \times 1 = 1
\end{aligned}$$

In the same way we obtain the rest of the rows of the matrix. which gives
$B = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$.

2. *Indirect method; we use the matrix A associated with φ in the canonical basis and the matrix P for the transition to the new basis.*

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, we obtain,

$$\begin{aligned} P^T AP &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix} = B \end{aligned}$$

If the given bilinear form is symmetric, then the associated matrix is symmetric in any basis, because for a basis $\{v_1, \cdots, v_n\}$ for E, we have

$$b_{ij} = \varphi(v_i, v_j) = \varphi(v_j, v_i) = b_{ji}$$

which allows only half of the coefficients to be calculated.

2.2 Congruent matrices

Definition 4 *Matrices that represent the same bilinear form in different bases are called congruent. We leave it to students to verify the following proposition:*

Proposition 5 *The relation "congruent to" in the set of square matrices is an equivalence relation. The equivalence classes of symmetric matrices are given by the classification of quadratic forms in the next chapter.*

2.3 Bilinear form and duality

Let $\varphi : E \times E \longrightarrow \mathbb{K}$ be a bilinear form. for every $y \in E$, the map

$$\begin{aligned} \varphi(\cdot, y) \quad &: \quad E \longrightarrow \mathbb{K} \\ x \quad &\mapsto \quad \varphi(x, y) \end{aligned}$$

is a linear form on $\mathbb{K}$, i.e. an element of the dual E^*. Therefore, for all $y \in E$, we can define a right linear map d_φ as follows:

$$\begin{aligned} d_\varphi \quad &: \quad E \longrightarrow E^* \\ y \quad &\mapsto \quad d_\varphi(y) = \varphi(\cdot, y) \end{aligned}$$

The linearity of d_φ follows directly from the linearity of φ. In the same way we define the linear application on the left. $g_\varphi(x) = \varphi(x, \cdot)$.

Definition 6 *The kernels of the applications d_φ and g_φ defined above are called right kernel and left kernel respectively. If the bilinear form φ is symmetric, then the right and left applications are the same and we denote them by Φ_φ, so the right kernel and the left kernel are the same and equal to* $\ker \Phi_\varphi$.

Remark 7 *From the above, for any bilinear form, the left and right kernels are given by*

$$\begin{aligned} \ker g_\varphi &= \{x \in E, \forall y \in E, \varphi(x, y) = 0\}, \\ \ker d_\varphi &= \{y \in E, \forall x \in E, \varphi(x, y) = 0\}. \end{aligned}$$

Let $A = (a_{ij})_{n\times n}$ the matrix associated with φ and x_1, ...,x_n, y_1, ..., y_n denote the coordinates of x and y in a given basis. Then, from the algebraic expression of φ, we have $\forall y \in E$,

$$\varphi(x, y) = (a_{11}x_1 + \cdots a_{n1}x_n)\, y_1 + \cdots (a_{1n}x_1 + \cdots a_{nn}x_n)\, y_n = 0$$

which is equivalent to

$$\left\{ \begin{array}{c} a_{11}x_1 + \cdots a_{n1}x_n = 0 \\ \vdots \\ a_{1n}x_1 + \cdots a_{nn}x_n = 0 \end{array} \right. \Leftrightarrow A^T x = 0 \Leftrightarrow x \in \ker A^T$$

So, looking for the kernel on the left is the same as looking for the kernel of the transpose of the matrix associated with φ. Proceeding in the same way, we have the kernel on the right is the kernel of the associated matrix.

In what follows we limit ourselves to symmetric bilinear forms.

2.4 The rank of a symmetric bilinear form, non-degenerate form

Definition 8 *If the kernel of the symmetric bilinear form φ is reduced to* $\{0\}$, *then, the form is said to be nondegenerate. The rank of the bilinear form φ is the rank of the application d_φ (which is also equal to the rank of g_φ), so, it is the rank of the matrix associated with φ in a basis of E.*

Note: Since E is finite-dimensional, the nondegenerate bilinear forms are those corresponding to invertible matrices, which is equivalent to d_φ is bijective, i.e. for any linear form $f \in E^*$, there exists a unique $y \in E$, such that $d_\varphi(y) = f$, such that $\forall x \in E$, $f(x) = \varphi(x, y)$.

2.5 The orthogonality for a symmetric bilinear form

Definition 9 *Let F be a vector subspace of E and φ a symmetric bilinear form on E. The orthogonal of F for φ is the set*

$$F^{\perp} = \{x \in E, \forall y \in F, \varphi(x, y) = 0\}$$

We leave it to the students to verify the following properties:

Proposition 10 *Let F be a vector subspace of E and φ a bilinear form on E.*

i) *$F^{\perp}$ is a vector subspace of E.*

ii) *$E^{\perp} = \ker \varphi \subset F^{\perp}$.*

iii) *$F \subseteq \left(F^{\perp}\right)^{\perp}$. Equality occurs if φ is non-degenerate.*

Theorem 11 *Let F be a vector subspace of E and φ a symmetric bilinear form on E. Then,*

$$\dim F^{\perp} = n - \dim F + \dim (F \cap \ker \varphi).$$

Proof. Let us take the application Φ_{φ} from E to E^{*} defined in the paragraph "Bilinear form and duality". Since F is a vector subspace of E, then $G = \Phi_{\varphi}(F)$ is a vector subspace of E^{*}. Hence,

$$\forall f \in G, \exists y \in F, \ f = \Phi_{\varphi}(y)$$

Now let us take the orthogonal of G for Φ_{φ}, we obtain

$$\begin{aligned} G^{\perp} &= \{x \in E, \forall f \in G, \ f(x) = 0\} \\ &= \{x \in E, \forall y \in F, (\Phi_{\varphi}(y))(x) = 0\} \\ &= \{x \in E, \forall y \in F, \varphi(x, y) = 0\} = F^{\perp} \end{aligned}$$

So according to dimension theorem for vector spaces, we get

$$\dim E = \dim E^{*} = \dim G + \dim G^{\perp} = \dim \Phi_{\varphi}(F) + \dim F^{\perp} \tag{2}$$

In the other hand, let $\Phi_{\varphi/F}$ be the restriction of Φ_{φ} to F and apply the dimension theorem, we have

$$\dim F = \dim \Phi_{\varphi/F}(F) + \dim \ker \Phi_{\varphi/F} \tag{3}$$

While

$$\begin{aligned} \Phi_{\varphi/F}(F) &= \Phi_{\varphi}(F) = G \\ \ker \Phi_{\varphi/F} &= \left\{x \in F, \Phi_{\varphi/F}(x) = 0\right\} \\ &= \{x \in F, \forall y \in E, \varphi(x, y) = 0\} = F \cap \ker \varphi \end{aligned}$$

So equality (3) becomes

$$\dim F = \dim G + \dim (F \cap \ker \varphi) \tag{4}$$

From Equalities (2), (3), (4), we have

$$\dim E = \dim F - \dim (F \cap \ker \varphi) + \dim F^{\perp}$$

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Corollary 12 *If φ is nondegenerate, then,* $\dim F^{\perp} = n - \dim F$.

3 Quadratic forms

Definition 13 *Let E be a vector space of dimension n over $\mathbb{K}$ and φ a symmetric bilinear form over E. We call the quadratic form q over E the application $q : E \longrightarrow \mathbb{K}$ defined by*

$$\forall x \in E,\ q(x) = \varphi(x, x)$$

The form is called real or complex according to $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The matrix associated with φ is called the matrix of q. The rank and the kernel of q are the rank and the kernel of that matrix. The quadratic form is called non-degenerate if φ is non-degenerate (i.e. the matrix is invertible)

3.1 Another equivalent definition of the quadratic form

Let E be a vector space over $\mathbb{K}$, equipped with a basis $\{v_1, \cdots, v_n\}$, φ a symmetric bilinear form over E and $A = (\varphi(v_i, v_j))_{n \times n}$ the matrix associated with φ in this basis. From the definition 13 and the algebraic expression of φ we have the following expression:

$$q(x) = \sum_{i,j=1}^{n} \varphi(v_i, v_j)\, x_i x_j$$

where $x = x_1 v_1 + \cdots + x_n v_n$. So we have the following equivalent definition:

Definition 14 *We call quadratic form q on E any homogeneous polynomial over $\mathbb{K}$* [1] *of degree two in the coordinates of x.*

In general, for $n = 2$, 3, or 4 we denote by (x, y), (x, y, z) or (x, y, z, t) for a vector X in the canonical basis of E.

3.2 Polar form of quadratic form

Lemma 15 *Let q be a quadratic form on the vector space E. The map $\varphi : E \times E \longrightarrow \mathbb{K}$, defined by*

$$\forall x,\ y \in E,\ \varphi(x, y) = \frac{1}{2}\left(q(x + y) - q(x) - q(y)\right)$$

is a symmetric bilinear form over E. It is called the polar form of q.

[1] a homogeneous polynomial, or algebraic form, is a polynomial in many indeterminates where all its non zero monomials have the same total degree. For example the polynomial $x^4 - 2x^3 y + x^2 y^2$ is homogeneous of degree 4.

Remark 16 *The polar form also can be given by*

$$\forall x,\ y \in E,\ \varphi(x,y) = \frac{1}{4}\left(q(x+y) - q(x-y)\right)$$

3.3 The parallelogram rule

It is easy to verify the following identity:

$$\forall x, y \in E, q(x+y) + q(x-y) = 2q(x) + 2q(y).$$

It is called the parallelogram rule. The identity is very important for normed spaces and it has its applications in functional analysis and Topology. [2]

3.4 Some important remarks

We leave to students to verfy the following properties:

- $\forall \lambda \in \mathbb{K},\ \forall X \in E,\ q(\lambda X) = \lambda^2 q(X)$.
- For any bilinear form φ, there exists a quadratic form q associated to the symmetric bilinear form φ_q defined by

$$\forall x,\ y \in E,\ \varphi_q(x,y) = \frac{\varphi(x,y) + \varphi(y,x)}{2} \tag{5}$$

- A bilinear form φ is said to be alternate if and only if the quadratic form q associated to φ_q vanishes..

  In fact,

$$\begin{aligned} \varphi \text{ is alternate} &\Leftrightarrow \forall x, y \in E, \varphi(x,y) = -\varphi(y,x) \\ &\Leftrightarrow \varphi(x,y) + \varphi(y,x) = 0 \\ &\Leftrightarrow \varphi_q = 0 \Rightarrow \forall x \in E, q(x) = \varphi_q(x,x) = 0 \end{aligned}$$

  Conversely, suppose that $\forall x \in E,\ q(x) = 0$, then

$$\begin{aligned} \forall x, y &\in E, 0 = q(x+y) = \varphi_q(x+y, x+y) \\ &= \varphi_q(x,y) + \varphi_q(y,x) = 2\varphi_q(x,y) \\ &= \varphi(x,y) + \varphi(y,x) \Rightarrow \varphi(x,y) = -\varphi(y,x). \end{aligned}$$

[2]

Theorem 17 *(Jordan-von Neumann Theorem) Let $(X, \|\cdot\|)$ be a generalized normed space. Then, there exists a scalar product $\langle \cdot,\cdot \rangle : X \times X \longrightarrow \mathbb{C}$, such that $\sqrt{\langle x, x\rangle} = \|x\|$ if and only if $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$, $\forall x,y \in X$.*

- The set $Q(E)$ of all quadratic forms defined on E is a $\mathbb{K}-$vector space. For every bilinear form φ, we associate a quadratic form q of φ_q defined in (5), there exists a linear map $\varphi \mapsto q$ from the vector space $B(E \times E)$ of bilinear forms to the vector space $Q(E)$, its kernel is constituted by the alternate bilinear forms. Lemma 15 assure the surjectivity of this map. Thus, according to the first theorem of the isomorphisms, $Q(E)$ may be identified as a subspace of symmetric bilinear forms, which gives in the end the following decomposition in direct sum

$$B(E \times E) = B(E \times E)_{sym} \oplus B(E \times E)_{alt},$$

where $B(E \times E)_{sym}$ and $B(E \times E)_{alt}$ designate the subspaces of symmetric bilinear forms and alternate bilinear forms respectively.

- The representation of q by the algebraic expression is equivalent to the following matrix representation:

$$q(x) = x^T A x$$

where $A = (\varphi(v_i, v_j))_{n \times n}$ is the associated matrix of q. If we change the basis, then, we obtain a new representation

$$q(x) = x^T B x,$$

but we always have $B = P^T A P$, where P is the passage matrix to the new basis.

- If f and g are two linear forms on E, then

$$\forall x \in E,\ q(x) = f(x)\, g(x)$$

is a quadratic form on E. In fact, a linear form is a sum of monomials of degree 1 in the coordinates of x. Thus, $f(x)\, g(x)$ is a sum of monomials of degree 2.

Question: Can we always decompose a quadratic form in a product of two linear forms? lthe answer is negative, it is sufficient to take the form q defined on $\mathbb{R}^2$ by

$$q(X) = x^2 + y^2$$

In general, on a field $\mathbb{K}$, every quadratic form $q(x) = \sum_{i=1}^{n} a_i x_i^2$ where at least a coefficient doesn't have a square root in $\mathbb{K}$ cannot be decomposed in a product of linear forms on $\mathbb{K}$. So, what are the conditions that should be verified by a quadratic form to be decomposed on a field?

Let φ be a symmetric bilinear form and f and g two linear forms such that

$$\varphi(x, y) = f(x)\, g(y)$$

Since the form is symmetric, then we have

$$\varphi(x, y) = g(y) f(x).$$

Thus, we notice that φ, f and g have the same kernel. If this kernel is the whole space, then, the form vanishes. Otherwise the kernel is an, hyperplan, which means that the equations which represent the hyperplan are equivalent. Thus, the linear forms f and g are proportional. Hence, we have the following proposition:

Proposition 18 *A quadratic form q can be decomposed in a product of two linear forms, if, and only if, the two forms are proportional. That gives $q(x) = \alpha(l(x))^2$ where $\alpha \in \mathbb{K}$, $l \in E^*$ and $\ker q = \ker l$.*

3.5 Examples of some quadratic forms

The following examples are real quadratic forms. We give some hints only, and we leave to students to verify that in practice.

1. $q : M_n(\mathbb{K}) \longrightarrow \mathbb{K}$, defined by:

$$q(A) = trace\left(A^T A\right)$$

   is a quadratic form on the matrix space $M_n(\mathbb{K})$. In fact, let us fined the algebraic expression of Q in the canonical basis of $M_n(\mathbb{K})$.

2. $q : M_2(\mathbb{K}) \longrightarrow \mathbb{K}$, defined by:

$$q(A) = \det A$$

   is a quadratic form. In fact, Prove that it is a homogenous polynomial of degree 2 in the coefficients of A.

3.6 Isotropic vectors for a quadratic form

Definition 19 *A non zero vector is called isotropic for a quadratic form q if it satisfies $q(x) = 0$. The set of isotropic vectors is called the isotropic cone.*

Remark 20
1. *The set of isotropic vectors is not forced to be a vector space, in general, it is the union of vector subspaces, and also, it contains $\ker q$. See exercises, 34, 37, 38, in the end of the chapter, where the set sometimes is vector space and sometimes is just the union of vector spaces.*

2. *Be careful!! the kernel can be a null space, but there exist the isotropic vectors.*

3.7 The orthogonality for a quadratic form

Definition 21 *The orthogonality for a quadratic form is the orthogonality for its polar form.*

Thus, the results mentioned in the related paragraph for symmetric bilinear forms are the same for this paragraph. See exercises, 34, 37, 38. Note that, the mentioned exercises become easier to solve after reading the next chapter, but students can still solve them using the elementary tools of factoring a polynomial of degree 2 into factors.

3.8 Definite positive quadratic form, negative definite, not defined, scalar product

Definition 22 *Let q be a quadratic form on E.*

i) *q is said to be defined on E if for any $x \in E$, $q(x) = 0 \Rightarrow x = 0$. Otherwise, the form is said to be not defined.*

ii) *q is said to be positive definite (positive) on E if, for any $x \in E$, $q(x) > 0$ ($q(x) \geq 0$).*

iii) *q is said to be negative definite (negative) on E if, for any $x \in E$, $q(x) < 0$ ($q(x) \leq 0$).*

iv) *The polar form is said to be positive definite, negative, not defined etc. according to its quadratic form. In particular, if the polar form positive definite, then, it is called the scalar product.*

Remark 23 1. *There exists an isotropic vector for $q \Leftrightarrow q$ is not defined.*

2. *The standard scalar product on $\mathbb{R}^n$ is the symmetric bilinear form φ written in the canonical basis of $\mathbb{R}^n$, defined by the identity matrix I_n, which gives*

$$\forall x = (x_1, ..., x_n), \ y = (y_1, ..., y_n) \in \mathbb{R}^n, \ \varphi(x, y) = \sum_{i=1}^{n} x_i y_i$$

often denoted by $\langle x; y \rangle$.

3.9 The quadratic space

Definition 24 *A quadratic space is a vector space E equipped with a quadratic form q, often denoted by (E, q). The space takes particular names according to the additional properties of the quadratic form. The quadratic space is called Euclidean or Hermitian if the polar form is a scalar product on $\mathbb{R}$ or $\mathbb{C}$. It is often called pre-hilbertian space..*

In the next chapter we will see that any scalar product is equivalent (congruent) to the standard scalar product, this is why we note for the space of a scalar product by $(E, \langle \cdot, \cdot \rangle)$.

4 Series of exercises

Exercise 25 *Let φ be a symmetric bilinear form on $\mathbb{R}^3$ given by the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

1. *What is the kernel and the rank of φ?*
2. *Determine a basis for the orthogonal for φ and compare them with the results of course on the dimension of the orthogonal of the following subspaces:.*

$$\begin{aligned} F &= vect\left\{v_1 = (1,0,1), v_2 = (0,1,-1)\right\}, \\ G &= vect\left\{u_1 = (1,0,1), v_2 = (1,0,0)\right\}, \\ W &= vect\left\{w_1 = (0,1,0), v_2 = (1,0,1)\right\}, \end{aligned}$$

Exercise 26 *Let $\mathbb{R}_2[X]$ be the polynomial vector space.*

1. *In the canonical basis of $\mathbb{R}_2[X]$, determine the associated matrix of the symmetric bilinear form φ defined by*

$$\varphi(f,g) = \int_0^1 f(t)\, g(t)\, dt$$

2. *Determine* $\ker \varphi$.
3. *The same question for the following form:*

$$\varphi(f,g) = f(0)\, g(0) + f(1)\, g(1).$$

Exercise 27 *Let $M_n(\mathbb{R})$ be the matrix vector space of order n.*

1. *Prove that the following map is a symmetric bilinear form on $M_n(\mathbb{R})$:*

$$\varphi(A,B) = trace\,(AB)$$

2. *Let us denote by $S_n(\mathbb{R}) \subset M_n(\mathbb{R})$, the subspace of symmetric matrices. Prove that the restriction of φ to $S_n(\mathbb{R}) \times S_n(\mathbb{R})$ is positive definite.*
3. *Determine the orthogonal of $S_n(\mathbb{R})$ for φ*

Exercise 28 *Determine the quadratic forms of the symmetric bilinear forms in the previous exercises.*

Exercise 29 *Let q be a definite quadratic form on E..*

1. *Prove that q is either negative definite, or positive definite (Indication: Suppose that there exist two vectors x and y such that $q(x) > 0$ and $q(y) < 0$ and take $f(t) = q(x + yt)$, then, find the roots of the quadratic equation $f(t) = 0$, and prove that there is an isotropic vector for q).*

2. *Now, we suppose that q is non degenerate but not definite. Prove that q hasn't a constant sign (Indication: Use the CAUCHY-SCHWARZ inequality).* [3]

Exercise 31 *Let Tr denoted the trace. Let q_1 and q_2 be two maps defined on $M_n(\mathbb{R})$ by $q_1(A) = (Tr(A))^2$ and $q_2(A) = Tr\left(A^T A\right)$. Prove that q_1 and q_2 are quadratic forms. Are they positive? positive definite?*

Exercise 32 *Let $E = \ell\left(\mathbb{R}^2\right)$, $(\lambda, \mu) \in \mathbb{R}^2$ and q defined on E by*

$$\forall u \in E,\ q(u) = \lambda Tr\left(u^2\right) + \mu \det u$$

1. *Verify that q is a quadratic form on E.*

2. *Determine the rank of q in function of λ and μ.*

3. *Determine the isotropic vectors of q in function of λ and μ.*

Exercise 33 *Let f_1, f_2,..., f_n be n continued functions on $[0,1]$. For $i, j = \overline{1,n}$, we put*

$$a_{i,j} = \int_0^1 f_i(t) f_j(t)\, dt \text{ and } \forall X = (x_1, ..., x_n) \in \mathbb{R}^n,\ q(X) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

1. *Prove that q is a positive quadratic form.*

2. *Prove that q is a definite positive quadratic form. if, and only if, the set $(f_1, ..., f_n)$ is free.*

3. *Write the associated matrix of q in the particular case $f_i(t) = t^{i-1}$ for $i = \overline{1,n}$.*

Exercise 34 *Lett $E = \mathbb{R}^3$ and q the map from E in $\mathbb{R}$ defined by :*

$$\forall X = (x, y, z) \in \mathbb{R}^3, q(X) = (x + y)^2 + 2(y - z)^2$$

1. *Prove that q is a quadratic form, and determine the associated matrix in the canonical basis.*

---

[3]

Lemma 30 *Let q be a positive quadratic form on $\mathbb{R}-$vector space E and φ its polar form. Then,*

$$\forall x, y \in E, |\varphi(x,y)| \leq \sqrt{q(x)}\sqrt{q(y)}.$$

This 'inequality is called CAUCHY-SCHWARZ 'inequality.

2. *Determine the orthogonal of* E *and deduce the rank of* q.
3. *Find the isotropic cone* $\Im$. *Prove that it is a vector subspace of* E.
4. *Prove that there exists only one vector subspace* F *of* E *totaly isotropic, i.e.* $\{0\} \neq F \subset F^{\perp}$ [4].
5. *Construct two vector spaces of* E, *isotropic*[5], *not totaly isotropic and of distinct dimensions.*

Exercise 37 *Lett* $E = \mathbb{R}^3$ *and* q *the map of* E *in* $\mathbb{R}$ *defined by :*

$$\forall X = (x, y, z) \in \mathbb{R}^3, q(X) = xy + yz$$

1. *Prove that* q *is a quadratic form, and determine the associated matrix in the canonical basis.*
2. *Determine the orthogonal of* E *and deduce the rank of* q.
3. *Find the isotropic cone* $\Im$, *and prove that it is not a vector subspace of* E.
4. *For any integer* p, $0 \leq p \leq 3$, *study the existence of a subspace of dimension* p, *totaly isotropic.*
5. *Construct two vector spaces of* E, *isotropic, not totaly isotropic and of distinct dimensions.*

Exercise 38 *Let* q *be the quadratic form on* $\mathbb{R}^3$ *defined by*

$$q(x, y, z) = x^2 + 3y^2 - 8z^2 - 4xy + 2xz - 10yz$$

1. *Determine* $\ker q$.
2. *Prove that the set of all isotropic vectors in* $\mathbb{R}^3$ *is the union of two vector planes and give their equations.*
3. *Calculate the orthogonal of the vector* $(1, 1, 1)$ *for* q.

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[4]

Theorem 35 *a subspace* F *is said to be totaly isotropic, if, and only if, it is a subset of the isotropic cone* $\Im$.

[5]

Definition 36 *We say that a subspace* G *is isotropic for a quadratic form* q, *if, and only if,* $G \cap G^{\perp} \neq \{0\}$, *where* $G^{\perp}$ *is the orthogonal of* G *for* q.