Chapter 1: Linear forms and duality

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1 Introduction

Recall that the set of linear maps of a vector space E into a vector space F on the same field $\mathbb{K}$ is a vector space over $\mathbb{K}$ denoted $\ell(E,F)$. It is of dimension $\dim E \times \dim F$ and isomorphic to the space of matrices $M_{\dim F \times \dim E}(\mathbb{K})$). Linear forms are special types of linear maps, some times, they are called covectors, as they are of great importance in the decomposition of quadratic forms into sums of squares, it is the representation of the quadratic form in the diagonal form.

Definition 1 *A linear form is a linear map of the vector space E into the body $\mathbb{K}$ (seen as a vector space on itself), its kernel is called a hyperplane.*

From the dimension theorem and the previous definition, we result that a linear form is either zero or surjective. In the second case, its kernel is supplementary to a vector line.

Example 2 *The trace is a linear form on the space of square matrices of order n. We deduce that the subspace of zero trace matrices is a hyperplane, hence the dimension is equal to $n^2 - 1$. Thus its supplementary is a subspace of scalar matrices.*

Definition 3 *The space of linear forms $\ell(E,\mathbb{K})$ is called the dual space of E, denoted by E^*.*

2 Matrix representation

Let $\{v_1, \cdots, v_n\}$ be basis for the vector space E, and $\varphi \in E^*$. Then, the matrix representing φ in this basis is a row matrix $1 \times n$ with coefficients $\varphi(v_i) \in \mathbb{K}$. In fact, let

$$x = x_1 v_1 + \cdots + x_n v_n \;\Rightarrow \varphi(x) = x_1 \varphi(v_1) + \cdots + x_n \varphi(v_n),$$

which gives in matrix form:

$$\varphi(x) = \begin{pmatrix} \varphi(v_1) & \cdots & \varphi(v_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

Thus we can conclude that every matrix of rank 1 can be identified to a linear form.

3 Dual basis and ante-dual basis

From Definition 3 and the matrix representation, one can easy see that the vector spaces E and E^* have the same dimension. Therefore, they are isomorphic.

Theorem 4 *For every basis $\{v_1, \cdots, v_n\}$ for the vector space E, there exists unique basis $\{\varphi_1, \cdots, \varphi_n\}$ for E^* satisfies the following condition: For every $i = 1, ..., n$, we have*

$$\varphi_i(v_j) = \delta_{ij}.$$

The basis $\{\varphi_1, \cdots, \varphi_n\}$ is called the dual basis of the basis $\{v_1, \cdots, v_n\}$, sometimes is denoted by $\{v_1^*, \cdots, v_n^*\}$.

Proof. From the matrix representation, it is clear that a linear form is entirely determined by the image of each vector from the basis $\{v_1, \cdots, v_n\}$. Thus for each fixed i, the n equations $\varphi_i(v_j) = \delta_{ij}$, $j = \overline{1, n}$ uniquely define the form φ_i.

Now, let us prove that $\{\varphi_1, \cdots, \varphi_n\}$ is a basis for E^*. Since E^* has the same dimension as E, it is sufficient to prove that the n forms are free:

Let α_1,..., $\alpha_n \in \mathbb{K}$, such that

$$\alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n = 0,$$

then, for $j = 1, ..., n$, we have

$$\begin{aligned} 0 &= (\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n)(v_j) \\ &= \alpha_1\varphi_1(v_j) + \cdots + \alpha_j\varphi_j(v_j) + \cdots + \alpha_n\varphi_n(v_j) \\ &= \alpha_1 \cdot 0 + \cdots + \alpha_j \cdot 1 + \cdots + \alpha_n \cdot 0 = \alpha_j \end{aligned}$$

(In the other words, for each v_i from the basis of E, we correspond the unique form φ_i from the basis of E^*). Therefore, for

$$x = x_1v_1 + \cdots + x_nv_n \in E$$

we have $v_i^*(x) = x_i$, which gives

$$x = v_1^*(x)\, v_1 + \cdots + v_n^*(x)\, v_n \tag{1}$$

i.e. the coordinates of a vector x in E in the given basis are the images of x by the dual basis. For $\varphi \in E^*$, such that

$$\varphi = \alpha_1 v_1^* + ... + \alpha_n v_n^*,$$

we have,

$$\varphi(x) = \alpha_1 v_1^*(x) + \cdots + \alpha_n v_n^*(x). \tag{2}$$

In the other hand, from equation (1), we have

$$\varphi(x) = v_1^*(x)\,\varphi(v_1) + \cdots + v_n^*(x)\,\varphi(v_n) \tag{3}$$

Since $\varphi(x)$ is uniquely represented, then Equations (2) and (3) give

$$\alpha_i = \varphi(v_i)\,, \text{ for } i = 1, ..., n.$$

Thus, we have the following corollary: ■

Corollary 5 *Let $\{v_1, \cdots, v_n\}$ be a basis for the vector space E and $\{v_1^*, \cdots, v_n^*\}$ be the dual basis for the dual vector space E^*. Then, the form- coordinates of a vector $x \in E$ and its dual $x^* = \varphi \in E^*$ in the related bases are*

$$x = \begin{pmatrix} v_1^*(x) \\ \vdots \\ v_n^*(x) \end{pmatrix}, \varphi = \begin{matrix} \varphi(v_1) & \cdots & \varphi(v_n) \end{matrix}$$

Example 6 *The canonical basis of the space of null trace matrices of order 2 is the following*

$$\left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Thus, the dual basis is $\{e_1^*, e_2^*, e_3^*\}$ *such that*

$$e_i^* \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & -a_{11} \end{pmatrix} = a_{1i}, \text{ for } i = 1, 2, 3.$$

Therefore, we can represent such matrix by form- coordinates in the canonical basis by the following column:

$$A = \begin{pmatrix} e_1^* (A) \\ e_2^* (A) \\ e_3^* (A) \end{pmatrix}$$

Also, we represent a linear form φ *on the space of null trace matrices in the dual basis by the row vector:*

$$\varphi = \begin{pmatrix} \varphi (e_1) & \varphi (e_2) & \varphi (e_3) \end{pmatrix}$$

For example, if $\varphi = trace$*, then,* $\varphi = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ *is the null form. It is the restriction of the trace on the space of square matrices to its kernel.*

Example 7 *Let* $A \in GL_n (\mathbb{K})$*, then the columns of* A *constitutes a basis for* $\mathbb{K}^n$*. The dual basis is given by the rows of its inverse.*

In fact, let $A = \begin{pmatrix} C_1 & \cdots & C_i & \cdots & C_n \end{pmatrix}$ *and* $A^{-1} = \begin{pmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix}.$

From equality $A^{-1}A = I_n$*, it deduced that* $L_i (C_j) = \delta_{ij}$*. i.e.* $L_i = C_i^*$*,* $i = 1, ..., n$*. Therefore the* R_i *constitute the dual basis. Hence, to find the ante-dual basis of the dual basis, we construct the row matrix of the given dual basis, then, we calculate its inverse. The columns of the inverse matrix constitute the ante-dual basis..*

4 The orthogonality with respect to the duality

Let F and F^* be two vector subspaces of E and E^* respectively. We let students to verify that the following sets are vector subspaces of E and E^* respectively:

$$\begin{aligned} F^{\perp} &= \{\varphi \in E^*, \forall v \in F, \varphi(v) = 0\} \\ (F^*)^{\perp} &= \{v \in E, \forall \varphi \in F^*, \varphi(v) = 0\} \end{aligned}$$

Definition 8 *Let F and F^* be two vector subspaces of E and E^* respectively. The space $F^{\perp}$ (resp. $(F^*)^{\perp}$) is called the orthogonal of F (resp. F^*) respected to the duality.*

The subspace of $\mathbb{K}^n$ of the solutions of an homogenous linear system is the orthogonal of the linear forms defining this system. For example, giving the following system:

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

Then $(F^*)^{\perp} = \{(3x, x, 5x, -5x), x \in \mathbb{R}\}$ is the orthogonal of $F^* = \{\varphi_1, \varphi_2, \varphi_3\}$, where the φ_i are the rows of the system matrix:

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that $\dim F^* + \dim (F^*)^{\perp} = \dim E = 4$. Thus, we can mention the following theorem:

Theorem 9 *Let F be a vector subspace of the vector space E. Then, the following relation holds:*

$$\dim F + \dim F^{\perp} = \dim E.$$

(The same property holds if we interchange E by E^*).

Indeed, theorem is an immediate result of the solutions of a linear system of p equations with n unknown coefficients. The space of solutions is of dimension equal to $n - p$.

5 Exercise series

Exercise 10 *(Lagrange Interpolation) Let $\mathbb{R}_n[X]$ be vector space of real polynomials of degree $\leq n$ and $a_0,\ldots,\ a_{n+1}$ distinct real numbers.*

1. *Prove that the set of polynomials $\{L_0, ..., L_n\}$ is a basis for $\mathbb{R}_n[X]$, where*
$$L_i = \prod_{j \neq i} \frac{X - a_j}{a_i - a_j},\ i = 0, \ldots, n$$

2. *Prove that the linear forms $P \mapsto P(a_i)$ for $i = 0,\ldots,\ n$ constitute a basis for $(\mathbb{R}_n[X])^*$, the dual of the basis $\{L_0, ..., L_n\}$.*

Exercise 11 *Let $a_0, \ .\ .\ .\ , a_n \in \mathbb{R}$ be distinct real numbers and $\varphi_0, \ .\ .\ .\ , \varphi_n$ be linear forms on $E = \mathbb{R}_n[X]$ given by the relations:*

$$\varphi_i(P) = P(a_i) \text{ for all } i = 0, ..., n.$$

1. *Prove that the set $\{\varphi_0, ..., \varphi_n\}$ is a dual basis for E and determine its ante-dual basis.*

2. *Deduce the same result for $\varphi_i(P) = P(i)$ for $i = 0, ..., n$.*

3. *Same question as in 1) for $n = 2$, where φ_i are defined by*
$$\varphi_i(P) = \int_0^1 x^i P(x)\, dx \text{ for all } i = 0, ..., n.$$

4. *Same question for $n = 2$ and the φ_i are defined by*
$$\varphi_0(P) = P(1),\ \varphi_1(P) = P'(1),\ \varphi_2(P) = \int_0^1 P(x)\, dx.$$

5.1 Supplementary exercise

Exercise 12 *Let $E = M_n(K)$; the vector space of matrices of order n with entries in a field $\mathbb{K}$..*

1. *Prove that the trace is a linear form on E.*

2. *Prove that the set of zero trace matrices is an hyperplane of the trace as a linear form.*

3. *Let φ be the endomorphism of E defined for every matrix $M \in E$, by*

$$\varphi(M) = tr(M) I_n + M$$

   Determine $tr(\varphi)$ and $\det(\varphi)$.

4. *(Questions 4 and 5 are linked) Let $A \in E$, prove that the map φ_A defined for every matrix $M \in E$, by $\varphi_A(M) = tr(AM)$ is a linear form.*

5. *Prove that every linear form φ is uniquely given by φ_A for $A \in E$.*

6 Solutions of the series

Solution of exercise 1

First of all, we notice that since the $a_0, \ . \ . \ . \ , a_n \in \mathbb{R}$ are all distinct, then the polynomials L_i are well defined and all distinct.

1. Since $|A| = n+1 = \dim E$, then to prove that the set of the polynomials

$$A = \{L_0, ..., L_n\}$$

is a basis for $E = \mathbb{R}_n[X]$, it is sufficient to prove that A is free. Let $\alpha_0, ..., \alpha_n \in \mathbb{R}$ such that

$$\alpha_0 L_0 + ... + \alpha_n L_n = 0.$$

That means the polynomial $\alpha_0 L_0 + ... + \alpha_n L_n$ is the zero polynomial. Therefore, for every $a_j \in \{a_0, ..., a_n\}$, the polynomial evaluated in a_j vanishes. It follows that

$$\alpha_0 L_0 (a_j) + ... + \alpha_n L_n (a_j) = 0.$$

By using the definition of L_i, that gives

$$L_i (a_j) = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for all } j \neq i \end{cases} .$$

Hence, for $j = 0$, we have

$$\alpha_0 L_0 (a_0) + ... + \alpha_n L_n (a_0) = 0 = \alpha_0 \times 1 + \alpha_1 \times 0 + ... + \alpha_n \times 0 = \alpha_0.$$

By the same manner we get

$$\alpha_1 L_1 (a_1) = 0 = \alpha_1, ..., \ \alpha_n L_n (a_n) = 0 = \alpha_n.$$

Consequently, the set A is free.

2. For all $i = 0, \ldots, n$, let φ_i be the linear forms defined by

$$\varphi_i : \underset{P \mapsto P(a_i)}{E \to E^*}.$$

Since $\dim E^* = \dim E = n+1$, then to prove that the set

$$A^* = \{\varphi_0, ..., \varphi_n\}$$

is a basis for E^*, it is sufficient to prove that it is free.

Let $\alpha_0, ..., \alpha_n \in \mathbb{R}$ such that

$$\alpha_0 \varphi_{_0} + ... + \alpha_n \varphi_n = 0.$$

That means the linear form

$$\varphi = \alpha_0 \varphi_0 + ... + \alpha_n \varphi_n$$

is the zero linear form. Therefore, for every $P \in \mathbb{R}_n[X]$, we get $\varphi(P) = 0$. It follows that

$$\alpha_0 \varphi_{_0}(P) + ... + \alpha_n \varphi_n(P) = 0 = \alpha_0 P(a_0) + ... + \alpha_n P(a_n).$$

Since $L_j \in \mathbb{R}_n[X]$, then, for $P = L_j$, the previous equality gives

$$0 = \alpha_0 L_j(a_0) + ... + \alpha_n L_j(a_n).$$

Since

$$L_j(a_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases},$$

then, for all $j = 0, ..., n$, we get

$$0 = \alpha_0 L_j(a_0) + ... + \alpha_n L_j(a_n) = \alpha_j L_j(a_j) = \alpha_j,$$

which means that the set $A^* = \{\varphi_0, ..., \varphi_n\}$ is free.

<u>Solution of exercise 2</u>

1. Let $\alpha_0, ..., \alpha_n \in \mathbb{R}$ such that

$$\alpha_0 \varphi_{_0} + ... + \alpha_n \varphi_n = 0.$$

Then, for every $P \in \mathbb{R}_n[X]$, we get $\varphi(P) = 0$. It follows that

$$\alpha_0 \varphi_{_0}(P) + ... + \alpha_n \varphi_n(P) = 0 = \alpha_0 P(a_0) + ... + \alpha_n P(a_n).$$

Therefore, for P is one of the elements of the canonical basis $\{1, X, ..., X^n\}$, we get

$$\varphi_i(P) = \varphi_i\left(X^j\right) = a_i^j \quad \text{for } j = 1, ..., n.$$

Thus we have the following system:

$$\begin{cases} \alpha_0 + \alpha_1 + ... + \alpha_n & = 0 \\ \alpha_0 a_0 + \alpha_1 a_1 + ... + \alpha_n a_n & = 0 \\ \vdots & \vdots \\ \alpha_0 a_0^n + \alpha_1 a_1^n + ... + \alpha_n a_n^n & = 0 \end{cases} \tag{4}$$

By putting the system in matrix form, we have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}$$

is a Vandermonde matrix, its determinant is

$$\det A = \prod_{j \neq i} (a_i - a_j).$$

Since all the a_i are distinct, then $\det A \neq 0$. Therefore System (4) has only zero as solution, i.e. $\alpha_i = 0$ for all $i = 1, ..., n$. Which means that the φ_i are linearly independent. Therefore the set $\{\varphi_0, ..., \varphi_n\}$ is a basis for $(\mathbb{R}_n[X])^*$.

The ante-dual basis is constituted of the polynomials P_0,..., P_n such that

$$\varphi_i(P_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases}$$

That gives

$$P_j(a_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases},$$

First method:

For all $j = 0, ..., n$, let P_j be as follows

$$P_j(X) = p_{0j} + ... + p_{jj}X^j + ... + p_{nj}X^n$$

Then for a fixed j and $i = 0, ..., n$, we have

$$\begin{aligned} P_j(a_0) &= 0 = p_{0j} + a_0 p_{1j} + a_0^2 p_{2j} + ... + a_0^n p_{nj} \\ P_j(a_1) &= 0 = p_{0j} + a_1 p_{1j} + a_1^2 p_{2j} + ... + a_1^n p_{nj} \\ &\vdots \\ P_j(a_j) &= 1 = p_{0j} + a_j p_{1j} + a_j^2 p_{2j} + ... + a_j^n p_{nj} \\ &\vdots \\ P_j(a_n) &= 0 = p_{0j} + a_n p_{1j} + a_n^2 p_{2j} + ... + a_n^n p_{nj} \end{aligned}$$

That means, for every fixed j, we have a linear system of the form

$$\begin{cases} p_{0j} + a_0 p_{1j} + a_0^2 p_{2j} + ... + a_0^n p_{nj} & = 0 \\ \vdots & \vdots \\ p_{0j} + a_j p_{1j} + a_j^2 p_{2j} + ... + a_j^n p_{nj} & = 1 \\ \vdots & \vdots \\ p_{0j} + a_n p_{1j} + a_n^2 p_{2j} + ... + a_n^n p_{nj} & = 0 \end{cases}$$

By putting the previous system in matrix form we have

$$\begin{pmatrix} 1 & a_0 & \cdots & a_0^n \\ 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} p_{0j} \\ \vdots \\ p_{jj} \\ \vdots \\ p_{nj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

The system matrix A is a Vandermonde matrix, which means that it is invertible, then the system has the unique solution

$$\begin{pmatrix} p_{0j} \\ \vdots \\ p_{jj} \\ \vdots \\ p_{nj} \end{pmatrix} = \begin{pmatrix} 1 & a_0 & \cdots & a_0^n \\ 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^n \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \tag{5}$$

That implies that the entries of each column of the inverse matrix represents the coefficients p_{ij} of corresponding polynomial P_j for all $j = 0, ..., n$. (for students: by using the matrix form in (5) determine the ante-dual basis for $n = 2$).

Second method:

Since

$$P_j(a_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases},$$

Then, for a fixed j and all $i = 0, ..., n$, with $i \neq j$, the a_i are roots of the polynomial P_j while a_j is not. That yields to

$$P_j(X) = \left(\prod_{i \neq j} (X - a_i)\right) Q_j(X) \tag{6}$$

and

$$P_j(a_j) = \left(\prod_{i \neq j} (a_j - a_i)\right) Q_j(a_j) = 1.$$

Therefore

$$Q_j(a_j) = \frac{1}{\prod_{i \neq j} (a_j - a_i)}.$$

That means that $Q_j(X)$ is a constant. Otherwise it would be a fraction, while $Q_j(X)$ is a polynomial.

By replacing the value of $Q_j(X)$ in relation (6), we get

$$P_j(X) = \frac{\prod_{i \neq j} (X - a_i)}{\prod_{i \neq j} (a_j - a_i)} = \prod_{i \neq j} \frac{X - a_i}{a_j - ai}.$$

Therefore, the ante-dual basis is constituted of Lagrange interpolations.

2. It is sufficient to take $a_i = i$ for all $i = 0, ..., n$, then

$$P_j(X) = \prod_{i \neq j} \frac{X - i}{j - i}.$$

3. For

$$\varphi_i(P) = \int_0^1 x^i P(x)\, dx.$$

Let $\alpha_0, ..., \alpha_n \in \mathbb{R}$ such that

$$\alpha_0 \varphi_0 + ... + \alpha_n \varphi_n = 0.$$

Then, for every $P \in \mathbb{R}_n[X]$, we get $\varphi(P) = 0$. It follows that

$$\alpha_0 \varphi_0(P) + ... + \alpha_n \varphi_n(P) = 0 = \alpha_0 \int_0^1 x^0 P(x)\, dx + ... + \alpha_n \int_0^1 x^n P(x)\, dx.$$

Therefore, for P is one of the elements of the canonical basis $\{1, X, ..., X^n\}$, we get the following system:

$$\begin{cases} \alpha_0 \int_0^1 X^0 dx + ... + \alpha_n \int_0^1 X^n dx & = 0 \\ \alpha_0 \int_0^1 X dx + ... + \alpha_n \int_0^1 X^{n+1} dx & = 0 \\ \vdots & \vdots \\ \alpha_0 \int_0^1 X^n dx + ... + \alpha_n \int_0^1 X^{2n} dx & = 0 \end{cases},$$

which gives

$$\begin{cases} \alpha_0 + \frac{1}{2}\alpha_1 + ... + \frac{1}{n+1}\alpha_n & = 0 \\ \frac{1}{2}\alpha_0 + \frac{1}{3}\alpha_1 + ... + \frac{1}{n+2}\alpha_n & = 0 \\ \vdots & \vdots \\ \frac{1}{n+1}\alpha_0 + \frac{1}{n+2}\alpha_1 ... + \frac{1}{2n+1}\alpha_n & = 0 \end{cases} \qquad (7)$$

System (7) in matrix form becomes

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix of the system is a Hilbert matrix $H = (h_{ij})$ of order $n+1$, such that the entries $h_{ij} = \frac{1}{i+j-1}$. The determinant of the Hilbert matrix is given by the following relation:

$$\det H = \frac{c_n^4}{c_{2n}},$$

where

$$c_n = \prod_{i=1}^{n-1} i^{n-i} = \prod_{i=1}^{n-1} i!.$$

That means $\det H \neq 0$. Therefore, System (7) has only zero as solution, i.e. $\alpha_i = 0$ for all $i = 1, ..., n$. Which means that the φ_i are linearly independent. Therefore the set $\{\varphi_0, ..., \varphi_n\}$ is a basis for $(\mathbb{R}_n [X])^*$.

Let us now find the ante-dual basis of $\{\varphi_0, ..., \varphi_n\}$. Let

$$P_j (X) = p_{0j} + p_{1j} X + ... + p_{nj} X^n, \text{ for all } j = 1, ..., n \tag{8}$$

such that

$$\varphi_i (P_j) = \int_0^1 X^i P_j (X)\, dx = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases} \tag{9}$$

Relations (8) and (9) yield to the following system in matrix form:

$$\begin{pmatrix} 1 & \frac{1}{2} & \cdots & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \frac{1}{j} & \frac{1}{j+1} & \cdots & \cdots & \frac{1}{n+j} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \cdots & \frac{1}{2n+1} \end{pmatrix} \begin{pmatrix} p_{0j} \\ \vdots \\ p_{jj} \\ \vdots \\ p_{nj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the matrix of the system is the Hilbert matrix, then it is invertible, which allows us to find all the polynomial P_j of the ante-dual basis by the relation:

$$\begin{pmatrix} p_{0j} \\ \vdots \\ p_{jj} \\ \vdots \\ p_{nj} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \frac{1}{j} & \frac{1}{j+1} & \cdots & \cdots & \frac{1}{n+j} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \cdots & \frac{1}{2n+1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Let the inverse matrix be $H^{-1} = \left(h'_{ij}\right)$

$$H^{-1} = \begin{pmatrix} h'_{11} & h'_{12} & \cdots & \cdots & h'_{1(n+1)} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h'_{j1} & h'_{j2} & \cdots & \cdots & h'_{j(n+1)} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h'_{(n+1)1} & h'_{(n+1)2} & \cdots & \cdots & h'_{(n+1)(n+1)} \end{pmatrix}.$$

Then,

$$\begin{pmatrix} h'_{1j} \\ \vdots \\ h'_{jj} \\ \vdots \\ h'_{nj} \end{pmatrix} = \begin{pmatrix} h'_{11} & h'_{12} & \cdots & \cdots & h'_{1(n+1)} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h'_{j1} & h'_{j2} & \cdots & \cdots & h'_{j(n+1)} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h'_{(n+1)1} & h'_{(n+1)2} & \cdots & \cdots & h'_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for all } j = 1, ..., n,$$

i.e. the elements of the ante-dual basis are the columns of the inverse of the system matrix.

For example, for $n = 2$, we have

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \ H^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}.$$

The elements of the ante-dual basis are:

$$P_0 \quad : \quad \begin{pmatrix} p_{00} \\ p_{10} \\ p_{20} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -36 \\ 30 \end{pmatrix}$$
$$P_1 \quad : \quad \begin{pmatrix} p_{01} \\ p_{11} \\ p_{21} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -36 \\ 192 \\ -180 \end{pmatrix}$$
$$P_2 \quad : \quad \begin{pmatrix} p_{02} \\ p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 30 \\ -180 \\ 180 \end{pmatrix}$$

Which means:

$$\begin{aligned} P_0(X) &= 30X^2 - 36X + 9 \\ P_1(X) &= -180X^2 + 192X - 36 \\ P_2(X) &= 180X^2 - 180X + 30 \end{aligned}$$

4. For

$$\varphi_0(P) = P(1), \; \varphi_1(P) = P'(1), \; \varphi_2(P) = \int_0^1 P(x)\,dx,$$

let $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_0 \varphi_0 + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 = 0.$$

Then, for every $P \in \mathbb{R}_2[X]$, we have

$$\begin{aligned} \alpha_0 \varphi_0(P) + \alpha_1 \varphi_1(P) + \alpha_2 \varphi_2(P) &= 0 \\ \alpha_0 P(1) + \alpha_1 P'(1) + \alpha_2 \int_0^1 P(X)\,dx &= 0 \end{aligned}$$

Therefore, for P is one of the elements of the canonical basis $\{1, X, X^2\}$, we get the following system:

$$\begin{cases} \alpha_0 + \alpha_2 & = 0 \\ \alpha_0 + \alpha_1 + \frac{1}{2}\alpha_2 & = 0 \\ \alpha_0 + 2\alpha_1 + \frac{1}{3}\alpha_2 & = 0 \end{cases} \tag{10}$$

The system matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{3} \end{pmatrix}$ is of determinant$= \frac{1}{3} \neq 0$, which means that the system has only zero solution. Therefore $\{\varphi_0, \varphi_1, \varphi_2\}$ is a basis for $(\mathbb{R}_2 [X])^*$. Now, let $\{P_0, P_1, P_2\}$ be the ante-dual basis for $(\mathbb{R}_2 [X])$. Then,

$$\varphi_i (P_j) = \varphi_i \left(p_{0j} + p_{1j} X + p_{2j} X^2\right) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for all } i \neq j \end{cases},$$

we get the following systems in matrix form

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} p_{00} \\ p_{10} \\ p_{20} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} p_{01} \\ p_{11} \\ p_{21} \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} p_{02} \\ p_{12} \\ p_{22} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{11}$$

$$P_0 \ : \ \begin{pmatrix} p_{00} \\ p_{10} \\ p_{20} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$P_1 \ : \ \begin{pmatrix} p_{01} \\ p_{11} \\ p_{21} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -2 \\ \frac{3}{2} \end{pmatrix}$$

$$P_2 \ : \ \begin{pmatrix} p_{02} \\ p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$$

Note that the matrix in Systems (11) is the transpose $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$ of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{3} \end{pmatrix}$ in System (10).

Its rows represent the dual basis $\{\varphi_0, \varphi_1, \varphi_2\}$, while the columns of its inverse $\begin{pmatrix} -2 & \frac{1}{2} & 3 \\ 6 & -2 & -6 \\ -3 & \frac{3}{2} & 3 \end{pmatrix}$ represent the ante-dual basis $\{P_0, P_1, P_2\}$:

$$\begin{aligned} P_0(X) &= -3X^2 + 6X - 2 \\ P_1(X) &= \frac{3}{2}X^2 - 2X + \frac{1}{2} \\ P_2(X) &= 3X^2 - 6X + 3 \end{aligned}$$

6.1 Solution of the supplementary exercise

1. Let $E = M_n(\mathbb{K})$, we have already shown that the trace is a linear form on E.

2. Since φ is a linear form, then for every $M \in M_n(\mathbb{K})$, it is a linear map with $\varphi(M) \in \mathbb{K}$. Then

$$\varphi(M) = 0 \Rightarrow M \in \ker \varphi,$$

and from the theorem of dimensions, we have

$$\varphi(M) \in \mathbb{K} \Rightarrow \dim \ker \varphi = n - 1.$$

That means $\ker \varphi$ is a hyperplane.

3. To determine $tr(\varphi)$ and $\det(\varphi)$, let us determine the eigenvalues of φ, then

$tr(\varphi) =$ the sum of the eigenvalues and $\det \varphi =$ the product of the eigenvalues

where for every $M \in M_n(\mathbb{K})$,

$$\varphi(M) = tr(M) I_n + M$$

For $E_{ij} \in M_n(K), i; j \in \{1, 2, ..., n\}$, we have

$$tr(E_{ij}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Thus,

$$\varphi(E_{ij}) = \begin{cases} I_n + E_{ij} & i = j \\ E_{ij} & i \neq j \end{cases}$$

Therefore, for every $i \neq j$, the matrix E_{ij} is an eigenvector of φ associated to the eigenvalue $\lambda_1 = 1$. Since we have n basis matrices E_{ii}, then, the number of basis matrices E_{ij} with $i \neq j$ is $n^2 - n$. Therefore, the eigenvalue $\lambda_1 = 1$ is of algebraic multiplicity $\geq n^2 - n$. For I_n, we have

$$tr(I_n) = n \Rightarrow \varphi(I_n) = nI_n + I_n = (n+1) I_n.$$

which means that I_n is an eigenvector of φ associated to the eigenvalue $\lambda_2 = n + 1$

For a diagonal matrix M with zero trace, we have $\varphi(M) = M$, that means every diagonal zero trace matrix is an eigenvector associated to the eigenvalue $\lambda = 1$. The space of diagonal matrices is of dimension n, because its canonical basis is the set $\{E_{ii} \in M_n(K), i \in \{1, 2, ..., n\}\}$. Therefore

$$tr(M) = 0 \Rightarrow M \in \ker tr$$

By the dimension theorem, it gives

$$\dim \ker tr = n - 1,$$

that means the space of diagonal zero trace matrices is of dimension $n - 1$. Therefore we have $((n^2 - n) + (n - 1)) = n^2 - 1$ eigenvectors associated to the eigenvalue $\lambda = 1$, thus, the eigenvalue $\lambda_1 = 1$ is of algebraic multiplicity $\geq n^2 - 1$.

Consequently, we have $\lambda_1 = 1$ eigenvalue of algebraic multiplicity $\geq n^2 - 1$ and $\lambda_2 = n + 1$ of algebraic multiplicity ≥ 1.

Since $\dim M_n(\mathbb{K}) = n^2 = (n^2 - 1) + 1$, that gives $\lambda_1 = 1$ eigenvalue of algebraic multiplicity $= n^2 - 1$ and $\lambda_2 = n + 1$ of algebraic multiplicity $= 1$. therefore

$$\begin{aligned} tr(\varphi) &= \left(n^2 - 1\right) \times 1 + (n + 1) = n^2 + n. \\ \det \varphi &= n + 1. \end{aligned}$$

4. For every M, $N \in M_n(\mathbb{K})$ and for every α, $\beta \in \mathbb{K}$, we have

$$\begin{aligned}\varphi_A(\alpha M + \beta N) &= tr(A(\alpha M + \beta N)) = tr((\alpha AM + \beta AN)) \\ &= \alpha tr(AM) + \beta tr(BM) = \alpha\varphi_A(M) + \beta\varphi_A(N)\end{aligned}$$

Thus, φ_A is a linear map. Since $\varphi_A(M) = tr(AM) \in \mathbb{K}$, then φ_A is a form. Therefore , φ_A is a linear form.

5. Let $\{E_{ij} \in M_n(\mathbb{K}), i, j \in \{1, 2, ..., n\}\}$ be the canonical basis of E. Then for every matrix $M = (m_{ij})_{n\times n}$, we have

$$M = \sum_{i;j=1}^{n} m_{ij}E_{ij}$$

Since φ is a linear form, we get

$$\varphi(M) = \sum_{i;j=1}^{n} m_{ij}\varphi(E_{ij}) \text{ and } \varphi(E_{ij}) \in \mathbb{K}$$

Set

$$\varphi(E_{ij}) = a_{ji} \text{ for all } i; j \in \{1, 2, ..., n\},$$

that gives a matrix $A = (a_{ji})_{n\times n}$, such that

$$\varphi(M) = \sum_{i;j=1}^{n} a_{ji}m_{ij} = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji}m_{ij}\right) \tag{12}$$

The expression in parentheses represents the matrix product AM. Let $AM = B = (b_{ji})_{n\times n}$, then the expression in parentheses becomes

$$a_{ji}m_{ij} = b_{jj}$$

Therefore the expression (12) becomes

$$\varphi(M) = \sum_{j=1}^{n} b_{jj} = tr(AM)$$

For uniqueness: let A, $B \in M_n(K)$, such that for every matrix $M = (m_{ij})_{n\times n}$, we have $tr(AM) = tr(BM)$, then

$$tr((A-B)M) = 0$$

Then for $M =$the transpose $(A-B)^t$, we get

$$tr\left((A-B)(A-B)^t\right) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|^2 = 0$$

which gives $a_{ij} - b_{ij} = 0$. Thus $A = B$.