2.4 Solution of exercises

Solution 2.3.1

1. f is linear if and only if
   $\forall\alpha,\beta\in\mathbb{R},\forall(x,y),(x',y')\in\mathbb{R}^2; f(\alpha(x,y)+\beta(x',y'))\overset{?}{=}\alpha f(x,y)+\beta f(x',y')$.

$$\begin{aligned}f(\alpha(x,y)+\beta(x',y')) &= f(\alpha x+\beta x',\alpha y+\beta y')\\&=(\alpha x+\beta x'+\alpha y+\beta y',\alpha x+\beta x'-\alpha y-\beta y')\\&=(\alpha x+\alpha y,\alpha x-\alpha y)+(\beta x'+\beta y',\beta x'-\beta y')\\&=\alpha(x+y,x-y)+\beta(x'+y',x'-y')\\&=\alpha f(x,y)+\beta f(x',y').\end{aligned}$$

   Hence, f is linear.
2. Let's determine $kerf$ and Imf and give their dimensions. Is f bijective ?
   $kerf=\{(x,y)\in\mathbb{R}^2\mid f(x,y)=(0,0)\}=\{(0,0)\}$, so $dim\,ker\,f=0$.
   $Imf=\{(x+y,x-y)\mid(x,y)\in\mathbb{R}^2\}=\{x(1,1)+y(1,-1)\mid(x,y)\in\mathbb{R}^2\}$.
   Thus, Imf is generated by two linearly independent vectors, so $dim\,Im\,f=2$.
   Since the dimension of the domain equals the dimension of the codomain, f is bijective as it is both injective and surjective. Specifically, f is injective because $kerf=\{(0,0)\}$, and it is surjective because $dim\,\mathbb{R}^2=dim\,Im\,f=2$.
3. For any $(x,y)\in\mathbb{R}^2$, we have
   $f\circ f(x,y)=f(f(x,y))=f(x+y,x-y)=(2x,2y)=2(x,y)=2\cdot\mathrm{Id}_{\mathbb{R}^2}$.

Solution 2.3.2

1. f is linear if and only if
   $\forall\alpha,\beta\in\mathbb{R},\forall(x,y),(x',y')\in\mathbb{R}^2; f(\alpha(x,y)+\beta(x',y'))\overset{?}{=}\alpha f(x,y)+\beta f(x',y')$.

$$\begin{aligned}f(\alpha(x,y)+\beta(x',y')) &= f(\alpha x+\beta x',\alpha y+\beta y')\\&=(2\alpha x+2\beta x'-4\alpha y-4\beta y',\alpha x+\beta x'-2\alpha y-2\beta y')\\&=(2\alpha x-4\alpha y,\alpha x-2\alpha y)+(2\beta x'-4\beta y',\beta x'-2\beta y')\\&=\alpha f(x,y)+\beta f(x',y').\end{aligned}$$

   Thus, f is linear.
2. Let's determine $kerf$ and Imf and give their dimensions. Is f bijective ?
   $kerf=\{(x,y)\in\mathbb{R}^2\mid f(x,y)=(0,0)\}=\{(2y,y)\mid y\in\mathbb{R}\}=\{y(2,1)\mid y\in\mathbb{R}\}$.
   So, $dim\,ker\,f=1$. Therefore, f is not injective.
   $Imf=\{(2x-4y,x-2y)\mid(x,y)\in\mathbb{R}^2\}=\{x(2,1)+y(-4,-2)\mid(x,y)\in\mathbb{R}^2\}$.
   Imf is generated by two dependent vectors since $(-4,-2)=-2(2,1)$. Hence, $dim\,Im\,f=1$. Alternatively, we can use the fact that $dim\,ker\,f+dim\,Im\,f=dim\,\mathbb{R}^2\Rightarrow dim\,Im\,f=2-1=1$.
3. f is not bijective as it is neither injective nor surjective.

Solution 2.3.3

1. Let $u=(x,y,z)$ and $u'=(x',y',z')$ be two vectors in $\mathbb{R}^3$, and let λ and λ' be real numbers.
   $\lambda u+\lambda'u'=(\lambda x+\lambda'x',\lambda y+\lambda'y',\lambda z+\lambda'z')$

$$\begin{aligned}f(\lambda u+\lambda'u')&=(\lambda x+\lambda'x'+\lambda y+\lambda'y'+\lambda z+\lambda'z',-(\lambda x+\lambda'x')+2(\lambda y+\lambda'y')+2(\lambda z+\lambda'z'))\\&=(\lambda(x+y+z)+\lambda'(x'+y'+z'),\lambda(-x+2y+2z)+\lambda'(-x'+2y'+2z'))\\&=\lambda(x+y+z,-x+2y+2z)+\lambda'(x'+y'+z',-x'+2y'+2z')\\&=\lambda f(u)+\lambda'f(u')\end{aligned}$$

   Therefore, f is linear.

2. $$\begin{aligned}\ker f &= \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z)=(0,0)\}\\ &= \{(x,y,z)\in\mathbb{R}^3 \mid (x+y+z,-x+2y+2z)=(0,0)\}\\ &\Rightarrow \begin{cases} x+y+z=0\\ 3y+3z=0\end{cases} \Rightarrow \begin{cases} x=0\\ y=-z\end{cases}\\ \ker f &= \{(0,-z,z)\mid z\in\mathbb{R}\}\\ &= \{z(0,-1,1)\mid z\in\mathbb{R}\}\end{aligned}$$

   Let $f=(0,-1,1)$; f is a basis for $\ker(f)$.

   $Imf=\{f(e_1),f(e_2),f(e_3)\}$

   $$\begin{cases} f(e_1)=(1,-1)=f_1-f_2\\ f(e_2)=(1,2)=f_1+2f_2\\ f(e_3)=(1,2)=f_1+2f_2\end{cases}$$

   $Imf=\{f_1-f_2,f_1+2f_2\}$

   f_1-f_2 and f_1+2f_2 are not proportional, so they form a linearly independent set in Imf. As they generate Imf, they form a basis for Imf, and thus $dim(Imf)=2$. Note that $Imf=\mathbb{R}^2$.

Solution 2.3.4

1. $$\begin{cases} f(e_1)=(1,2,0)=1\cdot e_1+2\cdot e_2+0\cdot e_3\\ f(e_2)=(0,1,-1)=0\cdot e_1+1\cdot e_2+(-1)\cdot e_3\\ f(e_3)=(-1,-3,2)=(-1)\cdot e_1+(-3)\cdot e_2+2\cdot e_3\end{cases}$$

2. — Coordinates of $f(e_1)$ in the basis (e_1,e_2,e_3) are $(1,2,0)$.
   — Coordinates of $f(e_2)$ in the basis (e_1,e_2,e_3) are $(0,1,-1)$.
   — Coordinates of $f(e_3)$ in the basis (e_1,e_2,e_3) are $(-1,-3,2)$.

3. $$(x,y,z)\in\ker f \Leftrightarrow \begin{cases} x-z=0\\ 2x+y-3z=0\\ -y+3z=0\end{cases} \Leftrightarrow \begin{cases} x-z=0\\ y-z=0\\ -y+3z=0\end{cases} \Leftrightarrow \begin{cases} x=0\\ y=0\\ z=0\end{cases}$$

   Therefore, $\ker f=\{0_{\mathbb{R}^3}\}$.

   First method :

   $Imf=\{f(e_1),f(e_2),f(e_3)\}$

   Check if the family $\{f(e_1),f(e_2),f(e_3)\}$ is linearly independent :

   $\alpha f(e_1)+\beta f(e_2)+\gamma f(e_3)=0_{\mathbb{R}^3}\Leftrightarrow \alpha\cdot(1,2,0)+\beta\cdot(0,1,-1)+\gamma\cdot(-1,-3,2)=(0,0,0)$

   $$\Rightarrow \begin{cases} \alpha-\gamma=0\\ 2\alpha+\beta-3\gamma=0\\ -\beta+2\gamma=0\end{cases}$$

   This is the same system as before, so $\alpha=\beta=\gamma=0$. The family is linearly independent and spans Imf, so it forms a basis for Imf, and $dim(Imf)=3$, concluding that $Imf=\mathbb{R}^3$.

   Second method (more complicated) :

   $$\begin{aligned} Imf=\{f(e_1),f(e_2),f(e_3)\} &= \{e_1+2e_2,e_2-e_3,-e_1-3e_2+2e_3\}\\ &= \{e_1+2e_2,e_2-e_3,-e_2+2e_3+e_1+2e_2\}\\ &= \{e_1+2e_2,e_2-e_3-e_2+2e_3,-e_2+2e_3\}\\ &= \{e_1+2e_2,e_3,-e_2+2e_3\}\\ &= \{e_1+2e_2,e_3,-e_2+2e_3-2e_3\}\\ &= \{e_1+2e_2,e_3,-e_2\}\\ &= \{e_1+2e_2,e_3,e_2\}\\ &= \{e_1,e_3,e_2\}\\ &= \{e_1,e_2,e_3\}\end{aligned}$$

   Thus, a basis for Imf is (e_1,e_2,e_3), and of course, $Imf=\mathbb{R}^3$.

   Third method :

   Using the rank-nullity theorem, $dim(\ker f)+dim(Imf)=dim(\mathbb{R}^3)=3$. Since $dim(\ker f)=0$, $dim(Imf)=3$, and therefore $Imf=\mathbb{R}^3$. A basis for Imf is $(f(e_1),f(e_2),f(e_3))$.

Solution 2.3.5

1. The matrix of $f \circ f$ in the basis β is $\mathcal{M}_\beta(f) \times \mathcal{M}_\beta(f)$, where
   $\mathcal{M}_\beta(f) = \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix}$.
   Therefore, $\mathcal{M}_\beta(f) \times \mathcal{M}_\beta(f) = \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Id_{\mathbb{R}^3}$.
2. There exists g such that $g \circ f = Id$, so f is bijective, and $f^{-1} = g = f$.