Chapter 1

Vector spaces

1.1 Vector Space and Subspace

Definition 1.1.1

Let K be a commutative field ($K = \mathbb{R}$ or $K = \mathbb{C}$), and let the non-empty set E be equipped with an internal operation denoted by $(+)$:

$$\begin{aligned}(+): \quad E \times E &\to E \\ (x, y) &\mapsto x + y,\end{aligned}$$

and also equipped with an external operation denoted by $(\,\cdot\,)$:

$$\begin{aligned}(\,\cdot\,): \quad \mathbb{K} \times E &\to E \\ (\lambda, x) &\mapsto \lambda \cdot x\end{aligned}$$

Definition 1.1.2

A vector space over the field $\mathbb{K}$ or an $\mathbb{K}$-vector space is a triplet $(E, +, \,\cdot\,)$ such that :

1. $(E, +)$ is a commutative group.
2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E, \lambda(x + y) = \lambda \cdot x + \lambda \cdot y$.
3. $\forall \lambda, \mu \in \mathbb{K},\ \forall x \in E,\ (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$.
4. $\forall \lambda, \mu \in \mathbb{K},\ \forall x \in E,\ (\lambda \cdot \mu) \cdot x = \lambda(\mu \cdot x)$.
5. $\forall x \in E,\ 1_{\mathbb{K}} \cdot x = x$.

The elements of K are called scalars, and those of the vector space are called vectors.

Proposition 1.1.3

If the set E is an $K-$vector space, then it satisfies the following properties :

1. $\forall x \in E,\ 0_{\mathbb{K}} \cdot x = 0_E$.
2. $\forall x \in E,\ -1_{\mathbb{K}} \cdot x = -x$.
3. $\forall \lambda \in \mathbb{K},\ \lambda \cdot 0_E = 0_E$.
4. $\forall \lambda \in \mathbb{K},\ \forall x, y \in E,\ \lambda \cdot (x - y) = \lambda \cdot x - \lambda \cdot y$.
5. $\forall \lambda \in \mathbb{K},\ \forall x \in E,\ x \cdot \lambda = 0_E \Leftrightarrow x = 0_E \vee \lambda = 0_{\mathbb{K}}$.

Example 1.1.4

1. $(\mathbb{R}, +, \cdot)$ is an $\mathbb{R}$-vector space, $(\mathbb{C}, +, \cdot)$ is a $\mathbb{C}$-vector space.

2. If we consider the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the following operations :

$$\begin{array}{rll} (+): & \mathbb{R}^2 \times \mathbb{R}^2 & \to \mathbb{R}^2 \\ & ((x,y),(x',y')) & \mapsto (x+x', y+y'). \end{array}$$

$$\begin{array}{rll} (\cdot): & \mathbb{R} \times \mathbb{R}^2 & \to \mathbb{R}^2 \\ & (\lambda,(x,y)) & \mapsto (\lambda \cdot x, \lambda \cdot y). \end{array}$$

So, $(\mathbb{R}^2, +, .)$ can be easily shown as an $\mathbb{R}-$vector space

Definition 1.1.5

Let F be a non-empty subset of E, and $(E, +, .)$ be an $K-$vector space. We say that F is a subspace if $(F,+, .)$ is also an $K-$vector space.

Note 1.1.6

— When $(F, +, \cdot)$ is an $\mathbb{K}$-subspace of $(E, +, \cdot)$, then $0_E \in F$.
— If $0_E \notin F$, then $(F, +, \cdot)$ cannot be an $\mathbb{K}$-subspace of $(E, +, \cdot)$.

Theorem 1.1.7

Let $(E, +, \cdot)$ be an $\mathbb{K}$-vector space and $F \subset E, F$ non-empty. The following are equivalent :

1. F is a subspace of E.
2. F is stable under addition and multiplication, i.e.,
$$\forall x, y \in F, \forall \lambda \in \mathbb{K}, x + y \in F, \lambda \cdot x \in F.$$
3. $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F$, i.e., F is a subspace $\Leftrightarrow \begin{cases} F \neq \phi, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F \end{cases}$
4. $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F$, i.e., F is a subspace $\Leftrightarrow \begin{cases} 0_E \in F, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F \end{cases}$

Example 1.1.8

1. $\{0_E\}$ and E are subspaces of E.
2. $F = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$ is a subspace because,
   (a) $0_E = 0_{\mathbb{R}^2} = (0, 0) \in F$, then $F \neq \phi$
   (b) $\forall (x, y), (x', y') \in F, \lambda, \mu \in \mathbb{R}$, we want to prove that $\lambda(x, y) + \mu(x', y') \in F$, i.e., $(\lambda x + \lambda y) + (\mu x' + \mu y') = 0$, which holds true.
   Therefore, $\lambda(x, y) + \mu(x', y') \in F$, and F is a subspace of $\mathbb{R}^2$.
3. The set represented by : $F = \{(x + y + z, x - y, z) \mid x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ because,
   (a) $0_{\mathbb{R}^3} = (0, 0, 0) \in F$ since $(0, 0, 0) = (0 + 0 + 0, 0 - 0, 0) \Rightarrow F \neq \phi$.
   (b) $\forall X, Y \in F, \lambda, \mu \in \mathbb{R}$, we want to prove that $\lambda X + \mu Y \in F$. Let $X = (x, y, z)$ and $Y = (x', y', z')$. Then,
$$\begin{aligned} \lambda X + \mu Y &= (\lambda x + \lambda y + \lambda z, \lambda x - \lambda y, \lambda z) + (\mu x' + \mu y' + \mu z', \mu x' - \mu y', \mu z'), \\ &= ((\lambda x + \lambda y + \lambda z) + (\mu x' + \mu y' + \mu z'), (\lambda x - \lambda y) + (\mu x' - \mu y'), \lambda z + \mu z'), \\ &= ((\lambda x + \mu x') + (\lambda y + \mu y') + (\lambda z + \mu z'), (\lambda x - \mu y') + (\mu x' - \mu y'), \lambda z + \mu z'). \end{aligned}$$
   Therefore, $\lambda X + \mu Y \in F$. So, F is a subspace of $\mathbb{R}^3$.

Theorem 1.1.9
The intersection of a non-empty families of subspaces are defined as the subspace.

Note 1.1.10
The union of two subspaces is not always a subspace.

Example 1.1.11
Let $E_1 = \{(x,0) \in \mathbb{R}^2\}$ and $E_2 = \{(0,y) \in \mathbb{R}^2\}$. The union $E_1 \cup E_2$ is not a subspace in light of $U_1 = (1,0) \in E_1$, $U_2 = (0,1) \in E_2$, but $U_1 + U_2 = (1,1) \notin E_1 \cup E_2$, as $(1,1) \notin E_1$ and $(1,1) \notin E_2$.

1.1.1 Sum of Two Subspaces

Definition 1.1.12
Let a vector space E consist of two subspaces, E_1 and E_2. The sum of the two vector spaces denoted by $E_1 + E_2$, is defined as follows :

$$E_1 + E_2 = \{U \in E \mid \exists U_1 \in E_1,\ \exists U_2 \in E_2 \text{ such that } U = U_1 + U_2\}.$$

Proposition 1.1.13
The sum of two subspaces E_1 and E_2 of the same vector space E is a subspace of E containing $E_1 \cup E_2$.

1.1.2 Direct Sum of Two Subspaces

Definition 1.1.14
The sum $E_1 + E_2$ is said to be direct if, for every $U = U_1 + U_2$, there exists a unique vector $U_1 \in E_1$ and a unique vector $U_2 \in E_2$ such that $U = U_1 + U_2$. This is denoted as $E_1 \oplus E_2$.

Theorem 1.1.15
The sum $E_1 + E_2$ is direct if and only if $E_1 \cap E_2 = \{0_E\}$.

1.1.3 Generating Sets, Linearly Independent Sets, Bases, and Dimension

In the following, we denote the vector space $(E, +, .)$ as E.

Definition 1.1.16
Let E be a vector space and $e_1, e_2, ..., e_n$ be elements of E.

1. The set $\{e_1, e_2, ..., e_n\}$ is called linearly independent if for all $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$:

$$\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n = 0 \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0,$$

otherwise, they are called dependent.

2. The set $\{e_1, e_2, ..., e_n\}$ is called a generating set of E or E is generated by $\{e_1, e_2, ..., e_n\}$ if for every $x \in E$, there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$.
3. If $\{e_1, e_2, ..., e_n\}$ is a linearly independent and generating set of E, then it is called a basis of E.

Note 1.1.17
In a vector space E, every nonzero vector is linearly independent.

Theorem 1.1.18
If $\{e_1, e_2, ..., e_n\}$ and $\{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ are two bases of the vector space E, then $n = m$. In other words, if a vector space has a basis, then all bases of E have the same number of elements (or the same cardinality), and this number depends only on the vector space E.

Definition 1.1.19
If E is an $\mathbb{R}$-vector space with basis $B = \{e_1, e_2, ..., e_n\}$, then the dimension of E is defined as :

$$\dim(E) = \text{Card}(B).$$

Note 1.1.20
Finding a set of vectors in E that form a linearly independent basis for a vector space is known as basis. and generating set for E. The number of elements in this set represents $\dim(E)$.

Example 1.1.21

1. Find a basis for $\mathbb{R}^3$. We can use the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
2. Show that $f_1 = (1, -1), f_2 = (1, 1)$ form a basis for $\mathbb{R}^2$. Show that $\{f_1, f_2\}$ is linearly independent and generating.
3. Consider $F = \{(x + y, x - z, -y - z) \mid x, y, z \in \mathbb{R}\}$. Find a basis for F.

1.1.4 Vector Space of Finite Dimension (Properties)

Theorem 1.1.22
Let E be a vector space of dimension n :

1. The set $\{e_1, e_2, ..., e_n\}$ is a basis of E if and only if $\{e_1, e_2, ..., e_n\}$ is generating and linearly independent.
2. If $\{e_1, e_2, ..., e_p\}$ is a set of p vectors in E with $p > n$, then $\{e_1, e_2, ..., e_p\}$ cannot be linearly independent. Moreover, if $\{e_1, e_2, ..., e_p\}$ is generating, then there exist n vectors among $\{e_1, e_2, ..., e_p\}$ that form a basis for E.
3. If $\{e_1, e_2, ..., e_p\}$ is a set of p vectors in E with $p < n$, then $\{e_1, e_2, ..., e_p\}$ cannot be generating. Moreover, if $\{e_1, e_2, ..., e_p\}$ is linearly independent, then there exist $(n - p)$ vectors among $\{e_{p+1}, e_{p+2}, ..., e_n\}$ in E such that $\{e_1, e_2, ..., e_p, e_{p+1}, e_{p+2}, ..., e_n\}$ is a basis for E.

Example 1.1.23

1. In the previous example, $f_1 = (1, -1)$, $f_2 = (2, 1)$ to show that $\{f_1, f_2\}$ forms a basis of $\mathbb{R}^2$, it suffices to show that $\{f_1, f_2\}$ is either linearly independent or generating (this property holds in the case of finite-dimensional vector spaces).
2. To show that $\{f_1 = (1, 1, 1),\ f_2 = (1, 1, 0),\ f_3 = (0, 1, -1)\}$ is a basis of $\mathbb{R}^3$, it suffices to show that it is either linearly independent or generating since $dim\ \mathbb{R}^3 = 3$. $\{f_1, f_2, f_3\}$ is linearly independent because : For all $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$:

$$\lambda_1(1, 1, 1) + \lambda_2(1, 1, 0) + \lambda_3(0, 1, -1) = (0, 0, 0) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0,$$

which implies that $\{(1,1,1),(1,1,0),(0,1,-1)\}$ is a basis of $\mathbb{R}^3$.

3. Let's find a basis for $F = \{(x+y, x-z, -y-z) \mid x,y,z \in \mathbb{R}\}$. Since $F \subset \mathbb{R}^3$, $dimF \leq 3$, so the basis of F does not have more than three vectors. $(x+y, x-z, -y-z) = x(1,1,0)+y(1,0,-1)+z(0,-1,-1)$. Thus, $v_1 = (1,1,0)$, $v_2 = (1,0,-1)$, $v_3 = (0,-1,-1)$ form a generating family for F. If this family is linearly independent, then it forms a basis for F.
   For all λ_1, λ_2 ,$\lambda_3 \in \mathbb{R}$,
   $$\lambda_1(1,1,0) + \lambda_2(1,0,-1) + \lambda_3(0,-1,-1) = (0,0,0) \Rightarrow \lambda_2 = -\lambda_1, \lambda_1 = \lambda_3.$$
   So, $\{(1,1,0),(1,0,-1),(0,-1,-1)\}$ is not linearly independent. However, according to the previous theorem, we can extract from this family a basis for F. To do this, we need to find two vectors from the family that are linearly independent. If found, they form a basis for F ; otherwise, we take a non-null vector, and it becomes the basis for F. For example, let's take $\{v_1, v_2\}$:
   $$\lambda_1(1,1,0) + \lambda_2(1,0,-1) = (0,0,0) \Rightarrow \lambda_1 = \lambda_2 = 0.$$
   Thus, $\{v_1, v_2\}$ is a basis for F and $dimF = 2$.

1.1.5 Supplementary Vector Subspace

Definition 1.1.24

Assume that vector space E has two subspaces, E_1 and E_2.
If $E_1 \oplus E_2 = E$, then E_1 and E_2 are considered supplemental.

Example 1.1.25

Let $E_1 = \{(x,0) \in \mathbb{R}^2\}$, and $E_2 = \{(0,y) \in \mathbb{R}^2\}$.
Note that : $E_1 \oplus E_2 = \mathbb{R}^2$, then E_1 and E_2 are supplementary.