Exercises on Functions and Limits

Exercise 1: Basic Limits

Evaluate the following limits:

1.
$$\lim_{x \to 3} (2x^2 - 5x + 1)$$

2.
$$\lim_{x \to -2} \frac{3x^2 - 4x + 1}{x + 2}$$

3.
$$\lim_{x \to 0} \frac{\sin(5x)}{x}$$

4.
$$\lim_{x \to \infty} \frac{5x^3 + 2x}{3x^3 - 4x^2 + 1}$$

Exercise 2: One-Sided Limits

Evaluate the left-hand and right-hand limits:

1.
$$\lim_{x \to 0^+} \frac{1}{x}$$

2.
$$\lim_{x \to 0^-} \frac{1}{x}$$

3.
$$\lim_{x \to 1^+} \frac{x - 1}{x^2 - 1}$$

4.
$$\lim_{x \to 1^-} \frac{x - 1}{x^2 - 1}$$

Exercise 3: Indeterminate Forms

Use L'Hôpital's Rule to evaluate the following indeterminate forms:

1. $$\lim_{x\to 0} \frac{e^x - 1}{x}$$

2. $$\lim_{x\to \infty} \frac{x}{\ln(x)}$$

3. $$\lim_{x\to 0} \frac{x - \sin(x)}{x^3}$$

4. $$\lim_{x\to 0} \frac{\tan(x) - x}{x^3}$$

Exercise 4: Continuity and Limits

Determine if the following functions are continuous at the specified points and justify your answer using limits:

1. $$f(x) = \frac{x^2 - 4}{x - 2}, \quad \text{at } x = 2$$

2. $$g(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}, \quad \text{at } x = 1$$

3. $$h(x) = \begin{cases} 3x + 1 & \text{if } x < 2 \\ x^2 - 4x + 5 & \text{if } x \geq 2 \end{cases}, \quad \text{at } x = 2$$

Exercise 5: Challenging Limits

1. Prove that:

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$$

2. Evaluate:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

3. Show that:

$$\lim_{x\to \infty} \left(x - \sqrt{x^2 + 2x}\right) = 1$$

4. Given $f(x) = \frac{\sin(x)}{x}$, prove that:

$$\lim_{x\to 0} f'(x) = 0$$

Exercise 6: Composite and Piecewise Functions

1. Let:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x + 1 & \text{if } x \geq 1 \end{cases}$$

Find the following:

$$\lim_{x \to 1^-} f(x), \quad \lim_{x \to 1^+} f(x)$$

Is the function continuous at $x = 1$?
2. Given:

$$g(x) = \sqrt{1 + x^2}$$

Find:

$$\lim_{x \to 0} \frac{g(x) - 1}{x^2}$$

Exercise 7: Limits at Infinity

Evaluate the following limits as $x \to \infty$:
1.

$$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 5}$$

2.

$$\lim_{x \to \infty} \frac{\ln(x)}{x}$$

3.

$$\lim_{x \to \infty} \frac{e^x}{x^n}, \quad n \text{ is a positive integer}$$

4. Evaluate:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Exercise 8: Application of Limits in Real-World Problems

1. A tank is being filled with water at a rate of $\frac{1}{t+1}$ liters per second, where t is the time in seconds. Find the limit of the filling rate as $t \to \infty$. What does this result mean physically?
2. A certain population of bacteria grows according to the function:

$$P(t) = 1000 \left(1 - e^{-0.1t}\right)$$

Find the population as $t \to \infty$. What is the limiting population?