University of Oum El Bouaghi — Academic year: 2024/2025
Introduction to topology — License 2 - Mathematics

Sheet of exercises N°1

Exercise 1 *We equip $X = \{a, b, c, d\}$ with the topology $\tau = \{\varnothing, X, \{a\}, \{d\}, \{a, d\}\}$.*

1. *What are the closed sets of the topological space (X, τ)?*
2. *Give the set of neighborhoods of each of the points a, b and c.*
3. *Determine the closure, the interior and the boundary of the subset $A = \{a, b, c\}$.*
4. *Specify the accumulation points and the isolated points of A.*

Exercise 2 *We equip $\mathbb{R}$ with the usual topology.*

1. *Describe the interior, the closure and the boundary of the set $A = \left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$.*
2. *Specify the accumulation points and the isolated points of A.*

Exercise 3 *In $\mathbb{R}$ equipped with the usual topology, consider the part*

$$A =]-\infty, -1[\cup \left\{0, \frac{\pi}{4}, \sqrt{3}\right\} \cup \left\{3 - \frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}.$$

1. *Determine $\mathring{A}$, $\overline{A}$ and ∂A.*
2. *Determine the accumulation points and the isolated points of A.*

Exercise 4 *On the interval $X = [0, 1[$, we introduce the family τ of sets of the form $[0, a[$ with $0 \leq a \leq 1$.*

$$\tau = \{[0, a[\ ;\ 0 \leq a \leq 1\}.$$

1. *Show that τ is a topology on X.*
2. *What are the closed sets of (X, τ)?*
3. *Determine the closure and the interior of the interval $I = \left[\frac{1}{4}, \frac{3}{4}\right]$ in (X, τ).*
4. *Verify that (X, τ) is not separated.*
5. *Let $(x_n)_n$ be a sequence of numbers in X that converges in the usual sense to $\frac{1}{2}$, i.e.*

   $$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N : \quad \left| x_n - \frac{1}{2} \right| < \varepsilon.$$

   Show that in (X, τ), $(x_n)_n$ converges to any number $\ell \in \left[\frac{1}{2}, 1\right[$.
6. *Show that $\mathbb{Q} \cap X$ is dense in (X, τ). Deduce that (X, τ) is separable.*

Exercise 5 ***Cofinite topology.***

1. *Let X be a nonempty set.*

   a) *Show that the family $\tau = \{\varnothing\} \cup \{\mathcal{O} \in \mathcal{P}(X)\ ;\ \mathcal{O}^c \text{ is finite}\}$ is a topology on X (called Cofinite topology on X).*

   b) *Show that if X is an infinite set, then the topological space (X, τ) is not separated.*

   c) *Show that if X is an infinite set, then any infinite subset of X is everywhere dense. Deduce that (X, τ) is separable.*

2. *We equip $\mathbb{R}$ with the cofinite topology τ.*

   a) *Determine the interior, the closure and the boundary of each of the following parts:*

$$\mathbb{N}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad \text{and} \quad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}.$$

   b) *Show that $\mathbb{N}$ does not admit any isolated point. Deduce the set of accumulation points of $\mathbb{N}$.*

   c) *Compare the topology τ with the usual topology τ_u of $\mathbb{R}$.*

3. *Let f be the map of $(\mathbb{R}, \tau)$ in $(\mathbb{R}, \tau_u)$ defined by*

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x^2, & x > 0. \end{cases}$$

   a) *Draw the representative curve of f.*

   b) *Show that f is not continuous at any point of $\mathbb{R}$.*

Exercise 6 *Show that in a separated space X, a point $x \in X$ is an accumulation point of a part A if and only if every neighborhood of x intersects A in an infinite number of points.*

Exercise 7 *We propose to show that a subspace of a separable topological space is not necessarily separable.*

Let X be an **uncountable infinite set** *and a be a fixed point of X.*

1. *Consider the family $\tau = \{\varnothing\} \cup \{B \in \mathcal{P}(X)\ ;\ a \in B\}$.*

   a) *Show that τ is a topology on X (called Particular point topology).*

   b) *Describe the closed sets of the space (X, τ).*

   c) *Deduce that the space (X, τ) is separable.*

2. *Let the subset $A = X \setminus \{a\}$. Show that the topology τ_A induced on A by τ is the discrete topology on A and explain why (A, τ_A) is not separable.*