Mathematics for Engineering Sciences

Chapter 01: Matrix Analysis

Chapter Objective: Define the concepts of:

- Matrix
- Operations on matrices

Introduction:

Numerical analysis, also known as computational mathematics or applied mathematics, is the field where algorithms are studied to solve mathematical analysis problems using arithmetic calculations. These problems are based on systems that involve matrices. This chapter focuses mainly on matrix analysis. In Chapter 1 we used matrices and vectors as simple storage devices. In this chapter matrices and vectors take on a life of their own. We develop the arithmetic of matrices and vectors. Much of what we do is motivated by a desire to extend the ideas of ordinary arithmetic to matrices. Our notational style of writing a matrix in the form $A = [a_{ij}]$ hints that a matrix could be treated like a single number. What if we could manipulate equations with matrix and vector quantities in the same way that we do equations with scalars? We shall see that this powerful idea gives us now methods for formulating and solving practical problems. In this chapter we use it to find effective methods for solving linear and nonlinear systems, solve problems of graph theory and analyze an important modeling tool of applied mathematics called a Markov chain

I. Matrix

I.1 Definition

A matrix is a rectangular array of elements with n rows and m columns. Let A be a matrix. We denote a_{ij} as the element in the i row and j column of matrix A. In general, the matrix A is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1m} \\ a_{21} & a_{22} \cdots & a_{2m} \\ & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nm} \end{bmatrix}$$

The matrix A is said to be of size $n \times m$, where:

- n : number of rows
- m: number of columns

I.2 Some types of matrices

- **Rectangular matrix**: The number of rows is different from the number of columns $n \neq m$.

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1m} \\ a_{21} & a_{22} \cdots & a_{2m} \\ & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nm} \end{bmatrix}$$

- **Square matrix:** The number of rows equals the number of columns $n = m$.

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

There are several types of square matrices:

- **Diagonal matrix**: $a_{ij} = 0$ *for* $i \neq j$.

$$A = \begin{bmatrix} a_{11} & 0 \cdots & 0 \\ 0 & a_{22} \cdots & 0 \\ & \vdots & \\ 0 & 0 \cdots & a_{nn} \end{bmatrix}$$

- **Symmetric or antisymmetric matrix:**
  - ✓ Symmetric if $a_{ij} = a_{ji}$
  - ✓ Antisymmetric if $a_{ij} = -a_{ji}$
- **Upper triangular matrix:** $a_{ij} = 0$ *for* $i > j$

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ & \vdots & \\ 0 & 0 \cdots & a_{nn} \end{bmatrix}$$

- **Lower triangular matrix:** $a_{ij} = 0$ *for* $i < j$.

$$A = \begin{bmatrix} a_{11} & 0 \cdots & 0 \\ a_{21} & a_{22} \cdots & 0 \\ & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

- **Singular matrix**: A matrix is singular if its determinant is zero.
- **Orthogonal matrix**: A matrix A is orthogonal if $A^{-1} = A^T$.
- **Positive definite matrix**: A matrix A is positive definite if the elements *(a_{ii})* on the main diagonal are non-zero, even during Gaussian elimination.

I.3 Special matrices

- **Transpose matrix**: The transpose of a matrix A^T is obtained by swapping its rows with its columns.
- **Conjugate matrix**: The conjugate of matrix A is denoted by $\bar{A}$, where the elements are conjugated.
- **Adjoint matrix**: The adjoint of matrix A is denoted by $A^* = \bar{A}^T$.
- **Hermitian matrix**: A matrix is Hermitian if $A = A^*$
- **Dominant diagonal matrix**: A matrix is said to have a dominant diagonal if $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|$
- **Strongly dominant diagonal matrix**: A matrix is strongly dominant if $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$.

II. Operations on Matrices

II.1 Sum and Difference of Matrices

How should we define addition or subtraction of matrices? We take a clue from elementary two- and three-dimensional vectors, such as the type we would encounter in geometry or calculus. There, in order to add two vectors, one condition has to hold: the vectors have to be the same size. If they are the same size, we simply add the vectors coordinate by coordinate to obtain a new vector of the same size, which is what the following definition does.

Definition 2.2. Matrix Addition and Subtraction Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. Then the *sum* of the matrices, denoted by $A + B$, is the $m \times n$ matrix defined by the formula

$$A + B = [a_{ij} + b_{ij}].$$

The *negative* of the matrix A, denoted by $-A$, is defined by the formula

$$-A = [-a_{ij}].$$

Finally, the *difference* of A and B, denoted by $A - B$, is defined by the formula

$$A - B = [a_{ij} - b_{ij}].$$

Notice that matrices must be the same size before we attempt to add them. We say that two such matrices or vectors are conformable for addition.

Example Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

Find $A + B$, $A - B$, and $-A$.

Solution. Here we see that

$$A + B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3-3 & 1+2 & 0+1 \\ -2+1 & 0+4 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}.$$

Likewise,

$$A - B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3--3 & 1-2 & 0-1 \\ -2-1 & 0-4 & 1-0 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$$

The negative of A is even simpler:

$$-A = \begin{bmatrix} -3 & -1 & -0 \\ --2 & -0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

II.2 Multiplication by a Scalar

The next arithmetic concept we want to explore is that of scalar multiplication. Once again, we take a clue from the elementary vectors, where the idea behind scalar multiplication is simply to "scale" a vector a certain amount by multiplying each of its coordinates by that amount, which is what the following definition says.

Definition 2.3. **Scalar Multiplication** Let $A = [a_{ij}]$ be an $m \times n$ matrix and c a scalar. The *product* of the scalar c with the matrix A, denoted by cA, is defined by the formula

$$cA = [ca_{ij}].$$

Recall that the default scalars are real numbers, but they could also be complex numbers.

Example Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad c = 3.$$

Find cA, $0A$, and $-1A$.

Solution. Here we see that

$$cA = 3\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 & 3\cdot 1 & 3\cdot 0 \\ 3\cdot -2 & 3\cdot 0 & 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix},$$

while

$$0A = 0\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

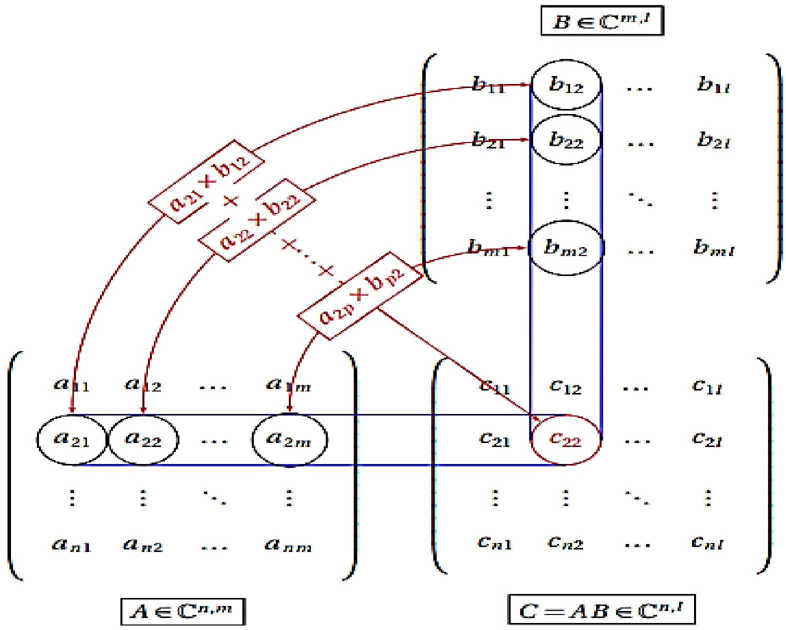
$$(-1)\,A = (-1)\begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -A.$$

II.3 Matrix Multiplication

If A has dimensions $n \times m$ and B has dimensions $m \times l$, the matrix product $C = A \times B$ is defined as a matrix of size $n \times l$, where:

$$c_{ij} = \sum_{k=1}^{m} a_{ik}\, b_{kj}$$

Note: The condition for the existence of the matrix product $A \times B$ is that the number of columns in A equals the number of rows in B.



Example . Compute, if possible, the products AB of the following pairs of matrices A, B.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution. First check conformability for multiplication. In part (a) A is 2×3 and B is 3×2. Stack these dimensions alongside each other and see that the 3's match; now "cancel" the matching middle 3's to obtain that the dimension of the product is $2\times \not{3}\ \not{3} \times 2 = 2 \times 2$. For example, multiply the first row of A by the second column of B to obtain the $(1,2)$th entry of the product matrix:

$$[1,2,1]\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = [1\cdot(-2) + 2\cdot 1 + 1\cdot 1] = [1] .$$

Similarly, the full product calculation looks like this:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 0 + 1\cdot 2 & 1\cdot(-2) + 2\cdot 1 + 1\cdot 1 \\ 2\cdot 4 + 3\cdot 0 + (-1)\cdot 2 & 2\cdot(-2) + 3\cdot 1 + (-1)\cdot 1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 1 \\ 6 & -2 \end{bmatrix} .$$

A size check of part (b) reveals a mismatch between the column number of the first matrix (3) and the row number (2) of the second matrix. Thus, these matrices are *not conformable* for multiplication in the specified order. Hence, the product

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

is undefined.

In part (c) a size check shows that the product has size $2\times \not{1}\ \not{1}\times 2 = 2\times 2$. The calculation gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 & 0\cdot 2 \\ 0\cdot 1 & 0\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For part (d) the size check shows gives $1\times \not{2}\ \not{2}\times 1 = 1\times 1$. Hence, the product exists and is 1×1. The calculation gives

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [1\cdot 0 + 2\cdot 0] = [0].$$

Matrix Multiplication Not Commutative or Cancellative

Something very interesting comes out of parts (c) and (d). Notice that AB and BA are *not* the same matrices—never mind that their entries are all 0's—the important point is that these matrices are not even the same size! Thus, a very familiar law of arithmetic, the commutativity of multiplication, has just fallen by the wayside.

Things work well in (e), where the size check gives $2\times \not{2}\ \not{2}\times 3 = 2\times 3$ as the size of the product. As a matter of fact, this is a rather interesting calculation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+0\cdot 2 & 1\cdot 2+0\cdot 3 & 1\cdot 1+0\cdot(-1) \\ 0\cdot 1+1\cdot 2 & 0\cdot 2+1\cdot 3 & 0\cdot 1+1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$$

Notice that we end up with the second matrix in the product. This is similar to the arithmetic fact that $1\cdot x = x$ for a real number x. So the matrix on the left acted like a multiplicative identity. We'll see that this is no accident.

Finally, for the calculation in (f), notice that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+1\cdot -1 & 1\cdot 1+1\cdot -1 \\ 1\cdot 1+1\cdot -1 & 1\cdot 1+1\cdot -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There's something very curious here, too. Notice that two nonzero matrices of the same size multiplied together to give a zero matrix. This kind of thing never happens in ordinary arithmetic, where the cancellation law assures that if $a \cdot b = 0$ then $a = 0$ or $b = 0$. □

The calculation in (e) inspires some more notation. The left-hand matrix of this product has a very important property. It acts like a "1" for matrix multiplication. So it deserves its own name. A matrix of the form

Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & \dots & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} = [\delta_{ij}]$$

is called an $n \times n$ *identity matrix*.

The (i, j)th entry of I_n is designated by the Kronecker symbol δ_{ij}, which is 1 if $i = j$ and 0 otherwise. If n is clear from context, we simply write I in place of I_n.

Kronecker Symbol

So we see in the previous example that the left-hand matrix of part (e) is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

III. Determinant

The determinant of a square matrix A is a scalar, denoted $det(A)$, and is defined as:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

III.1 Minor

For a square matrix A of order n, if we remove the $i - th$ row and the $j - th$ column, the determinant of the resulting matrix of order $n - 1$ is called the minor associated with the element a_{ij} of matrix A, denoted as m_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad ; \quad m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

III.2 Determinant Calculation

Definition 2.20. **Determinant** The *determinant* of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar quantity $\det A$ defined recursively as follows: If $n = 1$ then $\det A = a_{11}$; otherwise, we suppose that determinants are defined for all square matrices of size less than n and specify that

$$\det A = \sum_{k=1}^{n} a_{k1}(-1)^{k+1}M_{k1}(A)$$
$$= a_{11}M_{11}(A) - a_{21}M_{21}(A) + \cdots + (-1)^{n+1}a_{n1}M_{n1}(A),$$

where $M_{ij}(A)$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column of A.

Caution: The determinant of a matrix *A* is viewed as a scalar number, *not* a matrix.

Example . Use the definition to compute the determinants of the following matrices.

(a) $[-4]$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Solution. (a) From the first part of the definition we have $\det[-4] = -4$. For (b) we set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and use the formula of the definition to obtain that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a_{11}M_{11}(A) - a_{21}M_{21}(A) = a \det[d] - c \det[b] = ad - cb.$$

This calculation gives a handy formula for the determinant of a 2×2 matrix. For (c) use the definition to obtain that

$$\det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$
$$= 2(1 \cdot 2 - 1 \cdot (-1)) - 1(1 \cdot 2 - 1 \cdot 0) + 0(1 \cdot (-1) - 1 \cdot 0)$$
$$= 2 \cdot 3 - 1 \cdot 2 + 0 \cdot (-1)$$
$$= 4.$$

IV. Inverse Matrix

The inverse of a matrix A, denoted A^{-1}, is defined as the matrix such that:

$$A^{-1}.A = A.A^{-1} = I \text{ where } I \text{ is the identity matrix.}$$

> **Definition . Invertible Matrix** Let A be a square matrix. Then a *(two-sided) inverse* for A is a square matrix B of the same size as A such that $AB = I = BA$. If such a B exists, then the matrix A is said to be *invertible.*

Show that $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is an inverse for $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Solution. All we have to do is check the definition. But remember that there are two multiplications to confirm. (We'll show later that this isn't necessary, but right now we are working strictly from the definition.) We have:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 - 1 \cdot 1 & 2 \cdot 1 - 1 \cdot 2 \\ -1 \cdot 1 + 1 \cdot 1 & -1 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and similarly.

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot (-1) & 1 \cdot (-1) + 1 \cdot 1 \\ 1 \cdot 2 + 2 \cdot (-1) & 1 \cdot (-1) + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Therefore, the definition for inverse is satisfied, so that A and B work as inverses to each other.

Of course not every square matrix is invertible: Consider, e.g., zero matrices. However it is sometimes not entirely obvious why a matrix should not be invertible

IV.1 Laws of Inverses

Here are some of the basic laws of inverse calculations.

> **Laws of Matrix Inverses**
> Let A, B, C be matrices of the appropriate sizes so that the following multiplications make sense, I a suitably sized identity matrix, and c a nonzero scalar. Then
>
> (1) (Uniqueness) If the matrix A is invertible, then it has only one inverse, which is denoted by A^{-1}.
> (2) (Double Inverse) If A is invertible, then $\left(A^{-1}\right)^{-1} = A$.
> (3) (2/3 Rule) If any two of the three matrices A, B, and AB are invertible, then so is the third, and moreover, $(AB)^{-1} = B^{-1}A^{-1}$.
> (4) If A is invertible and $c \neq 0$, then $(cA)^{-1} = (1/c)A^{-1}$.
> (5) (Inverse/Transpose) If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$.
> (6) (Cancellation) Suppose A is invertible. If $AB = AC$ or $BA = CA$, then $B = C$.
> (7) (Rank) If A is invertible, then rank $A = n$ and the reduced row echelon form of A is I_n.

Note IV.1. Observe that the 2/3 Rule reverses order when taking the inverse of a product. This should remind you of the operation of transposing a product. A common mistake is to forget to reverse the order. Secondly, notice that the cancellation law restores something that appeared to be lost when we first discussed matrices. Yes, we can cancel a common factor from both sides of an equation, but (1) the factor must be on the same side and (2) the factor must be an invertible matrix.

Verification of Laws: Suppose that both B and C work as inverses to the matrix A. We will show that these matrices must be identical. The associative and identity laws of matrices yield

$$B = BI = B(AC) = (BA)C = IC = C.$$

Henceforth, we shall write A^{-1} for the unique (two-sided) inverse of the square matrix A, provided of course that there a ***Matrix Inverse Notation*** is an inverse at all (remember that existence of inverses is not a sure thing). The double inverse law is a matter of examining the definition of inverse:

$$AA^{-1} = I = A^{-1}A$$

shows that A is an inverse matrix for A^{-1}. Hence, $(A^{-1})^{-1} = A$. Now suppose that A and B are both invertible and of the same size. Using

the laws of matrix arithmetic, we see that

$$AB(B^{-1}A^{-1}) = A(BB^{-1})\, A^{-1} = AIA^{-1} = AA^{-1} = I$$

and that

$$(B^{-1}A^{-1})\, AB = B^{-1}(A^{-1}A)\, B = B^{-1}IB = B^{-1}B = I.$$

In other words, the matrix $B^{-1}A^{-1}$ works as an inverse for the matrix AB, which is what we wanted to show. We leave the remaining cases of the 2/3 Rule as an exercise.

Suppose that c is nonzero and perform the calculation:

$$(cA)(1/c)\, A^{-1} = (c/c)\, AA^{-1} = 1 \cdot I = I$$

A similar calculation on the other side shows that $(cA)^{-1} = (1/c)\, A^{-1}$. Next, apply the transpose operator to the definition of inverse and use the law of transpose products to obtain that

$$(A^{-1})^T . A^T = I^T = I = A^T . (A^{-1})^T$$

This shows that the definition of inverse is satisfied for $(A^{-1})^T$ relative to A^T , that is, that $(A^{-1})^T$=$(A^{-1})^T$, which is the inverse/transpose law. The same argument works with conjugate transpose in place of transpose.

Next, if A is invertible and $AB = AC$, then multiply both sides of this equation on the left by A^{-1} to obtain that

$$A^{-1}(AB) = (A^{-1}A)\ B = B = A^{-1}\ (AC) = (A^{-1}A)\ C = C$$

which is the cancellation that we want

We can now extend the power notation to negative exponents. Let A be an invertible matrix and k a positive integer. Then we write **Negative Matrix Power**

$$A^{-k} = A^{-1}A^{-1} \cdots A^{-1}.$$

where the product is taken over k terms.

The laws of exponents that we saw earlier can now be expressed for arbitrary integers, *provided* that A is invertible. Here is an example of how we can use the various laws of arithmetic and inverses to carry out an inverse calculation.

Example. Let: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Show that $(I - A)^3 = 0$ and use this to find A^{-1}.

Solution. First we calculate that

$$(I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and check that

$$(I - A)^3 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next, we do some symbolic algebra, using the laws of matrix arithmetic:

$$0 = (I - A)^3 = (I - A)\ (I^2 - 2AI + A^2) = I - 3A + 3A^2 - A^3.$$

Subtract all terms involving A from both sides to obtain that

$$3A - 3A^2 + A^3 = A \cdot 3I - 3A^2 + A^3 = A\ (3I - 3A + A^2) = I.$$

Since $A(3I - 3A + A^2) = (3I - 3A + A^2)A$, we see from definition of inverse that

$$A^{-1} = 3I - 3A + A^2 = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that in the preceding example we were careful not to leave a "3" behind when we factored out A from $3A$. The reason is that $3+3A+A^2$ makes no sense as a sum, since one term is a scalar and the other two are matrices.

The inverse of a matrix A can be calculated as:

$$A^{-1} = \frac{1}{\det(A)} Adj(A)$$

where $Adj(A)$ is the transpose of the cofactor matrix S.

The cofactor matrix S of a square matrix A is obtained by multiplying the minors by the sign matrix (or cofactor matrix). It is written as:

$$S = \begin{bmatrix} S_{11} & S_{12} \cdots & S_{1n} \\ S_{21} & S_{22} \cdots & S_{2n} \\ & \vdots & \\ S_{n1} & S_{n2} \cdots & S_{nn} \end{bmatrix}$$ where S_{ij}. is the cofactor of a_{ij}.

Inverse Algorithm

Given an $n \times n$ matrix A, to compute A^{-1}:

(1) Form the superaugmented matrix $\widetilde{A} = [A \mid I_n]$.
(2) Reduce the first n columns of $\widetilde{A}$ to reduced row echelon form by performing elementary operations on the matrix $\widetilde{A}$ resulting in the matrix $[R \mid B]$.
(3) If $R = I_n$ then set $A^{-1} = B$; otherwise, A is singular and A^{-1} does not exist.

Example Use the inverse algorithm to compute the inverse of Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. Notice that this matrix is already upper triangular. Therefore, as in Gaussian elimination, it is a bit more efficient to start with the bottom pivot and clear out entries above in reverse order. So we compute

$$[A \mid I_3] = \begin{bmatrix} 1&2&0&1&0&0 \\ 0&1&1&0&1&0 \\ 0&0&1&0&0&1 \end{bmatrix} \xrightarrow[E_{23}(-1)]{} \begin{bmatrix} 1&2&0&1&0&0 \\ 0&1&0&0&1&-1 \\ 0&0&1&0&0&1 \end{bmatrix} \xrightarrow[E_{1,2}(-2)]{} \begin{bmatrix} 1&0&0&1&-2&2 \\ 0&1&0&0&1&-1 \\ 0&0&1&0&0&1 \end{bmatrix}.$$

We conclude that A is indeed invertible and

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

V. Rank of a Matrix

The rank of a matrix A, denoted $r(A)$ or $rg(A)$, is the maximum number of linearly independent rows (or columns).

$$rg(A) \leq Inf(n,m)$$

VI. Trace of a Matrix

The trace of a square matrix A, denoted $tr(A)$, is the sum of the elements on its main diagonal:

$$tr\ (A) = \sum_{i=1}^{n} a_{ii}$$

VII. Elementary Matrix Transformations

VII.1 Definitions

An elementary matrix transformation refers to one of the following operations on the rows (or columns) of a matrix:

1. Swapping two rows (or columns)
2. Multiplying a row (or column) by a scalar
3. Adding d times one row (or column) to another row (or column), where $i \neq l$ and d is a scalar.

VII.2 Perlis Elementary Operations

Let *I* be the identity matrix. The following matrices are elementary matrices of Perlis:

- E_{ij} : Identity matrix I with the $i-th$ and $j-th$ rows swapped.
- $E_i(d)$: Identity matrix I with the $i-th$ row multiplied by d.
- $E_{il}(d)$: Identity matrix I with d times the l$j-th$ row added to the $i-th$ row.

The inverses of these matrices are:

- ${E_{ij}}^{-1} = E_{ij}$
- $E_i(d)^{-1} = E_i(1/d)$
- $E_{il}(d)^{-1} = E_i(-d)$.

VII.3 Elementary Transformations

Elementary transformations on a matrix A can be reduced to the pre-multiplication of A by one of the Perlis elementary matrices:

- $A_1 = E_{ij} \cdot A$ swaps the $i - th$ and $j - th$ rows.
- $A_2 = E_i(d) \cdot A$ multiplies the $i - th$ row of A by d.
- A_3=$E_i(-d). \cdot A$ adds d times the $l - th$ row of A to the $i - th$ row.