Chapter 2

Sets and Applications

2.1 Sets

Definition 2.1.1

A set is a collection of elements, for example $\{0, 1\}$, $\mathbb{N}$,

1. The empty set is a set containing no elements, denoted $\emptyset$.
2. We write $x \in E$ if x is an element of E, and $x \notin E$ otherwise.

2.1.1 Operations on Sets

Definition 2.1.2 Inclusion $F \subset E$

If every element of F is an element of E. In other words : $\forall x \in F$, $x \in E$. F is called a subset of E (or a part of E).

Definition 2.1.3 Equality

$$E = F \Leftrightarrow E \subset F \text{ and } F \subset E.$$

Definition 2.1.4 Power Set of E

We denote by $P(E)$ the power set of E. For example, if $E = \{1, 2, 3\}$. Then,

$$P(E) = \{\emptyset, E, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

If $card(E) = n$, then $card(P(E)) = 2^n$.

Definition 2.1.5 Difference and Symmetric Difference

Let A and B be two subsets of a set E. We denote :

1. The difference of A and B as the set :

$$A \setminus B = \{x \in A / x \notin B\}.$$

2. The symmetric difference of A and B as the set :

$$A \Delta B = (A \cup B) \setminus (A \cap B).$$

Definition 2.1.6 Complement of a Set

Let $A \subset E$. Then, the complement of A in E is denoted $\complement_E A$, which is defined by :

$$\complement_E A = \{x \in E / x \notin A\}.$$

It is also denoted $E \setminus A$ or A^c, or $\overline{A}$.

Definition 2.1.7 Intersection and Union

1. The intersection of A and B, denoted $A \cap B$, is the set of elements belonging to both A and B.
2. The union of A and B, denoted $A \cup B$, is the set of elements belonging to either A or B.

Formally, we have :

$$A \cap B = \{x / (x \in A) \wedge (x \in B)\}.$$

$$A \cup B = \{x / (x \in A) \vee (x \in B)\}.$$

Definition 2.1.8 Cartesian Product

The Cartesian product of sets A and B is the set of pairs $(x; y)$ where $x \in A$ and $y \in B$.

$$A \times B = \{(x; y) / x \in A \text{ and } y \in B\}.$$

If $card(A) = n$, $card(B) = m$. Then, $card(A \times B) = nm$.

Proposition 2.1.9

Let A, B, C be subsets of E. Then,

1. $A \cap B = B \cap A$, $A \cup B = B \cup A$;
2. $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$;
3. $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup \emptyset = A$, $A \cup A = A$;
4. $A \cap B = A \Leftrightarrow A \subset B$,
5. $A \cup B = B \Leftrightarrow A \subset B$;
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
8. $\complement_E(\complement_E A) = A$;
9. $\complement_E(A \cap B) = \complement_E A \cup \complement_E B$;
10. $\complement_E(A \cup B) = \complement_E A \cap \complement_E B$;
11. $A \subset B \Leftrightarrow \complement_E B \subset \complement_E A$.

Proof

8) Let $x \in E$. Then,

$$x \in \complement_E(\complement_E A) \Leftrightarrow x \notin \complement_E A \Leftrightarrow x \in A.$$

Thus,

$$\complement_E(\complement_E A) = \overline{\overline{A}} = A.$$

9) Let $x \in E$. Then,

$$x \in \complement_E(A \cap B) \Leftrightarrow x \notin (A \cap B) \Leftrightarrow (x \notin A) \vee (x \notin B) \Leftrightarrow (x \in \complement_E A) \vee (x \in \complement_E B) \Leftrightarrow x \in (\complement_E A \cup \complement_E B).$$

Thus,

$$\complement_E(A \cap B) = \complement_E A \cup \complement_E B.$$

Similarly, we prove property (10).
11) $A \subset B \Leftrightarrow \forall x \in E, ((x \in A) \Rightarrow (x \in B)) \Leftrightarrow \forall x \in E, ((x \notin B) \Rightarrow (x \notin A))$ (Contrapositive of the

implication)

$$\Leftrightarrow \forall x \in E, (x \in \complement_E B) \Rightarrow (x \in \complement_E A) \Leftrightarrow \complement_E B \subset \complement_E A,$$

thus

$$A \subset B \Leftrightarrow \complement_E B \subset \complement_E A.$$

2.2 Applications

Definition 2.2.1

A mapping or a function $f : E \to F$ is a relation that associates with each element $x \in E$ a unique element of F denoted $f(x)$.

1. f and g are two mappings. $f = g$ if and only if for all $x \in E$, $f(x) = g(x)$.
2. The graph of the mapping $f : E \to F$ is the set denoted G_f defined by

$$G_f = \{(x, f(x)) \in E \times F \ / \ x \in E\}.$$

3. The composition of two mappings f and g such that $f : E \to F$ and $g : F \to G$ is the mapping $g \circ f : E \to G$ defined by :

$$(g \circ f)(x) = g(f(x)).$$

Example 2.2.2

Let $f :]0, +1[\to]0, +1[$ and $g :]0, +1[\to \mathbb{R}$ be defined as

$$f(x) = \frac{1}{x},$$

and

$$g(x) = \frac{x-1}{x+1},$$

respectively :

$$g \circ f :]0, +1[\to \mathbb{R}$$
$$x \mapsto g(f(x)),$$

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x}\right), \\ &= \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1}, \\ &= \frac{1-x}{1+x}, \\ &= -g(x) \end{aligned}$$

Note 2.2.3

The composition of two mappings is not always defined. For example, $g \circ f$ is defined if the codomain of f is the same as the domain of g.

2.2.1 Direct Image, Inverse Image

Definition 2.2.4

1. Let $A \subset E$, and $f : E \to F$ be a mapping, the direct image of A under f is the set

$$f(A) = \{f(x)/x \in A\} \subset F,$$

i.e.,

$$y \in f(A) \Leftrightarrow \exists x \in A, y = f(x).$$

2. Let $B \subset F$, and $f : E \to F$ be a mapping, the inverse image of B under f is the set

$$f^{-1}(B) = \{x \in E/f(x) \in B\} \subset E,$$

i.e.,

$$x \in f^{-1}(B) \Leftrightarrow f(x) \in B.$$

Example 2.2.5

Let

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = x^2. \end{aligned}$$

Then,

$$f(\{2\}) = \{4\}.$$

$$f([-1,3]) = \{f(x)/x \in [-1,3]\} = [0,9].$$

$$f([-1,0] \cup [1,3]) = [0;9].$$

$$f^{-1}(\{2\}) = \{\ x \in \mathbb{R} \ \ / \ \ f(x) \in \{2\}\} = \left\{-\sqrt{2},\ \sqrt{2}\right\}.$$

Note 2.2.6

1. $f(A)$ is a subset of F, $f^{-1}(B)$ is a subset of E.
2. The notation $f^{-1}(B)$ does not imply that f is bijective, the inverse image exists for any function.
3. The direct image of a singleton $f(\{x\}) = \{f(x)\}$ is a singleton, whereas the inverse image of a singleton $f^{-1}(\{y\})$ depends on f, it can be a singleton, a set with multiple elements, or even E if f is a constant function.

Proposition 2.2.7

Let $f : E \to F$ be a mapping, A, A' subsets of E, and B, B' subsets of F.

1. $A \subset A' \Rightarrow f(A) \subset f(A^{'})$.
2. $B \subset B' \Rightarrow f^{-1}(B) \subset f^{-1}(B^{'})$.
3. $f(A \cap A') \subset f(A) \cap f(A^{'})$.
4. $f(A \cup A') = f(A) \cup f(A^{'})$.
5. $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B^{'})$.
6. $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B^{'})$.
7. $A \subset f^{-1}(f(A))$.
8. $f(f^{-1}(B)) \subset B$.

Proof

1) Suppose $y \in f(A)$. Then $\exists x \in A$ such that $y = f(x)$. Since

$$A \subset A', x \in A'.$$

Then,

$$y \in f(A').$$

Which implies

$$f(A) \subset f(A').$$

2) Let $x \in f^{-1}(B)$. Then, $f(x) \in B$. Since

$$B \subset B', f(x) \in B'.$$

Hence

$$x \in f^{-1}(B').$$

So,

$$f^{-1}(B) \subset f^{-1}(B^{'}).$$

3) Suppose $y \in f(A \cap A')$. Then, there exists

$$x \in (A \cap A')$$

such that $y = f(x)$. Since

$$x \in A, y = f(x) \in f(A),$$

and similarly

$$x \in A',$$

implies

$$y \in f(A').$$

Thus,

$$y \in f(A) \cap f(A^{'}).$$

So,

$$f(A \cap A') \subset f(A) \cap f(A^{'}).$$

4) Suppose

$$y \in (A \cup A') \exists x \in A \cup A',$$

such that

$$y = f(x).$$

If $x \in A$, then $y \in f(A)$ and if

$$x \in A'.$$

Then,

$$y \in f(A^{'}),$$

in both cases

$$y \in f(A) \cup f(A^{'}).$$

Hence,

$$f(A \cup A') \subset f(A) \cup f(A^{'}).$$

Conversely, if

$$y \in f(A) \cup f(A^{'}).$$

Then, if $y \in f(A)$ there exists $x \in A$ such that

$$y = f(x),$$

or if

$$y \in f(A').$$

Then, there exists

$$x \in A',$$

such that

$$y = f(x),$$

in both cases

$$y \in f(A \cup A').$$

Therefore,

$$f(A) \cup f(A^{'}) \subset f(A \cup A').$$

By mutual inclusion, we have equality.
5) Proven similarly to (3).
6) Proven similarly to (5).
7) Suppose $x \in A$. Let $B = f(A)$. Then $f(x) \in B$. So,

$$x \in f^{-1}(B) = f^{-1}(f(A)).$$

Hence $A \subset f^{-1}(f(A))$.
8) Suppose $y \in f(f^{-1}(B))$. Let $A = f^{-1}(B)$. Then $y \in f(A)$ implies

$$\exists x \in A,\ y = f(x).$$

Since,

$$x \in A = f^{-1}(B).$$

We have $f(x) \in B$, thus $y \in B$. Which implies

$$f(f^{-1}(B)) \subset B.$$

Definition 2.2.8 Antecedent

1. Let $y \in F$, any element $x \in E$ such that $f(x) = y$ is called an antecedent of y.
2. In terms of inverse image, the set of antecedents of y is $f^{-1}(\{y\})$.

2.2.2 Injection, surjection, bijection

Definition 2.2.9

Let $f : E \to F$ be a function :

1. f is injective if every element of the codomain has at most one pre-image under f.
2. f is surjective if every element of the codomain has at least one pre-image under f.
3. f is bijective if every element of the codomain has exactly one pre-image under f.

This definition can be reformulated as

Definition 2.2.10

1. f is injective if for every $y \in F$, the equation $f(x) = y$ has at most one solution in E.
2. f is surjective if for every $y \in F$, the equation $f(x) = y$ has at least one solution in E.
3. f is bijective if for every $y \in F$, the equation $f(x) = y$ has exactly one solution in E.
   Alternatively, f is bijective if it is injective and surjective.

Proposition 2.2.11

Let $f : E \to F$ be a function, the following statements are equivalent :

1. f is injective $\Leftrightarrow \forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
2. f is injective $\Leftrightarrow \forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
3. f is surjective $\Leftrightarrow \forall y \in F\,;\, \exists x \in E, y = f(x)$.
4. f is surjective $\Leftrightarrow f(E) = F$
5. f is bijective $\Leftrightarrow \forall y \in F\,;\, \exists! x \in E, y = f(x)$. The symbol ! denotes uniqueness, i.e., there exists a unique solution for the equation $f(x) = y$.

Example 2.2.12

Let the functions

$$\begin{aligned} f_1 : \quad & \mathbb{N} \to \mathbb{R} \\ & x \mapsto f_1(x) = \frac{1}{1+x}. \end{aligned}$$

$$\begin{aligned} f_2 : \quad & \mathbb{R}^+ \to \mathbb{R} \\ & x \mapsto f_2(x) = x^2. \end{aligned}$$

$$\begin{aligned} f_3 : \quad & \mathbb{R} \to \mathbb{R}^+ \\ & x \mapsto f_3(x) = x^2. \end{aligned}$$

Are the functions f_1, f_2, f_3 injective, surjective, bijective ?

Proof

1)

$$\begin{aligned} f_1 : \quad & \mathbb{N} \to \mathbb{R} \\ & x \mapsto f_1(x) = \frac{1}{1+x}. \end{aligned}$$

1. $$\forall x_1, x_2 \in \mathbb{N}, \quad f_1(x_1) = f_1(x_2) \Rightarrow \frac{1}{1+x_1} = \frac{1}{1+x_2} \Rightarrow x_1 = x_2.$$

   So f_1 is injective.

2. $$\forall y \in \mathbb{R}, \quad \frac{1}{1+x} = y \Rightarrow x = \frac{1-y}{y}.$$

   For example for $y = 5$, we get
   $$x = \frac{-4}{5} \notin \mathbb{N}.$$
   So, f_1 is not surjective. Therefore, f_1 is not bijective.

2)

$$\begin{aligned} f_2 : \quad & \mathbb{R}^+ \to \mathbb{R}, \\ & x \mapsto f_2(x) = x^2. \end{aligned}$$

1. $$\forall x_1, x_2 \in \mathbb{R}^+, f_2(x_1) = f_2(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow x_1 = x_2,$$

   (because $x_1, x_2 \in \mathbb{R}^+$). So f_2 is injective.

2. $$\forall y \in \mathbb{R}, \quad x^2 = y \Rightarrow x = \pm\sqrt{y}.$$

If $y \geq 0$. Then, for

$$y \in \mathbb{R}^- \nexists x \in \mathbb{R}^+.$$

Hence, f_2 is not surjective.
Therefore, f_2 is not bijective.

3)

$$\begin{array}{rccl} f_3: & \mathbb{R} & \to & \mathbb{R}^+ \\ & x & \mapsto & f_3(x) = x^2. \end{array}$$

1.

$$\forall x_1, x_2 \in \mathbb{R}, \quad f_3(x_1) = f_3(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2.$$

$$\exists 2, -2 \in \mathbb{R}, 2 \neq -2.$$

But $2^2 = (-2)^2$. So, f_3 is not injective.

2.

$$\forall y \in \mathbb{R}^+, x^2 = y \Rightarrow x = \pm\sqrt{y}.$$

If $y \geq 0$. Then,

$$\forall y \in \mathbb{R}^+, \exists x \in \mathbb{R}, y = f_3(x).$$

Hence f_3 is surjective.

Therefore, f_3 is not bijective.

Proposition 2.2.13

Let $f : E \to F$ and $g : F \to G$, then

1. f injective and g injective $\Rightarrow g \circ f$ injective,
2. f surjective and g surjective $\Rightarrow g \circ f$ surjective,
3. $g \circ f$ injective $\Rightarrow f$ injective,
4. $g \circ f$ surjective $\Rightarrow g$ surjective.

Proof

1. Let $x_1, x_2 \in E$. Then

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

because f is injective.

$$\Rightarrow g(f(x_1)) \neq g(f(x_2)),$$

because g is injective

$$\Rightarrow g \circ f(x_1) \neq g \circ f(x_2),$$

which shows that $g \circ f$ is injective.

2. Let $z \in G$. Since g is surjective, there exists $y \in F$ such that $z = g(y)$. We have $y \in F$ and f is surjective. Then, there exists $x \in E$ such that $y = f(x)$. Hence $z = g(f(x))$ and we conclude that :

$$\forall z \in G, \quad \exists x \in E, z = g \circ f(x).$$

It shows that $g \circ f$ is surjective.

3.

$$\forall x_1, x_2 \in E, \quad f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)).$$

Because g is a function.

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2).$$

$$\Rightarrow x_1 = x_2.$$

Because $g \circ f$ is injective, hence f is injective.

4. Let $z \in G$. Then, $g \circ f$ surjective

$$\Rightarrow \exists x \in E, \quad g \circ f(x) = z.$$

$$\Rightarrow \exists x \in E, \quad g(f(x)) = z.$$

$$\Rightarrow \exists y = f(x) \in F, \quad g(y) = z.$$

Thus,

$$\forall z \in G \, \exists y \in F, \quad g(y) = z.$$

Which shows that g is surjective.

2.2.3 Inverse function

Proposition 2.2.14

An application $f : E \to F$ is bijective if and only if there exists a unique function $g : F \to E$ such that

$$f \circ g = \mathrm{Id}_F \quad \text{and} \quad g \circ f = \mathrm{Id}_E.$$

We say that f is invertible and g, denoted f^{-1}, is called the “inverse function” or “reciprocal function” of f.

Proof

1. Suppose there exists a function $g : F \to E$ such that $f \circ g = \mathrm{Id}_F$ and $g \circ f = \mathrm{Id}_E$.
   Let's show that f is bijective.
   (a) Let $y \in F$. Since
   $$f \circ g = \mathrm{Id}_F.$$
   Then,
   $$f \circ g(y) = y.$$
   Thus there exists
   $$x = g(y) \in E,$$
   such that $f(x) = y$, showing that f is surjective.
   (b) Let $x_1, x_2 \in E$. Since $g \circ f = \mathrm{Id}_E$. Then,
   $$g \circ f(x_1) = x_1,$$
   and
   $$g \circ f(x_2) = x_2.$$
   Hence,
   $$f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)).$$
   Because g is a function.
   $$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2).$$
   $$\Rightarrow x_1 = x_2,$$
   showing that f is injective. From (1) and (2), we deduce that f is bijective.
2. Suppose f is bijective. Let's construct the unique function $g : F \to E$, such that $f \circ g = \mathrm{Id}_F$ and $g \circ f = \mathrm{Id}_E$.
   Since f is bijective, then for every $y \in F$, there exists a unique $x \in E$ such that $y = f(x)$.
   Thus, to every element $y \in F$, we associate a unique element $x \in E$, denoted by $g(x)$, such that $f(x) = y$. We define an application as follows :
   $$\begin{aligned} g &: F \to E \\ y &\mapsto g(y) = x \end{aligned}$$
   Let's show that
   $$f \circ g = \mathrm{Id}_F, \quad \text{and} \quad g \circ f = \mathrm{Id}_E.$$

(a) Let $y \in F$. Then $g(y) = x$, with $f(x) = y$. So,

$$f \circ g(y) = f(g(y)) = f(x) = y.$$

Showing that :

$$f \circ g = \mathrm{Id}_F.$$

(b) Let $x \in E$. Then for $y = f(x)$ we have $g(y) = x$. Thus,

$$g \circ f(x) = g(f(x)) = g(y) = x.$$

Which shows that :

$$g \circ f = \mathrm{Id}_E.$$

(c) Let's show the uniqueness of g. Let $g_1 : F \to E$ satisfying the two previous properties. Then, for every $y \in F$, there exists $x \in E$ such that $y = f(x)$. Thus

$$g_1(y) = g_1(f(x)) = g_1 \circ f(x) = \mathrm{Id}_E(x) = g \circ f(x) = g(f(x)) = g(y),$$

which shows that $g_1 = g$.

Example 2.2.15

$f : \mathbb{R} \to]0, +\infty[$ defined by

$$f(x) = \exp(x) = e^x,$$

is bijective. its inverse function is $g :]0, +\infty[\to \mathbb{R}$ defined by $g(y) = \ln(y)$. We indeed have

$$e^{\ln(y)} = y, \forall\ y\ \in]0, +\infty[, \text{ and } \ln(e^x) = x, \forall x \in \mathbb{R}.$$

Proposition 2.2.16

Let $f : E \to F$ and $g : F \to G$ be bijective applications. The function $g \circ f$ is bijective and its inverse is :

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof

According to proposition 2.5, there exists $u : F \to E$ such that $u \circ f = \mathrm{Id}_E$ and $f \circ u = \mathrm{Id}_F$.
There, also exists $v : G \to F$ such that $v \circ g = \mathrm{Id}_F$ and $g \circ v = \mathrm{Id}_G$.
Then $(g \circ f) \circ (u \circ v) = g \circ (f \circ u) \circ v = g \circ \mathrm{Id}_F \circ v = g \circ v = \mathrm{Id}_E$.
Also, $(u \circ v) \circ (g \circ f) = u \circ (v \circ g) \circ f = u \circ \mathrm{Id}_F \circ f = u \circ f = \mathrm{Id}_E$.
So $g \circ f$ is bijective and its inverse is $u \circ v$.
Since u is the inverse of f and v is the inverse of g, then : $u \circ v = f^{-1} \circ g^{-1}$.

2.2.4 Extension and Restriction

Definition 2.2.17

Let $f : E \to F$ be an application, let $A \subset E$; $B \subset F$ such that $f(A) \subset B$. We call the restriction of f to A as the starting set and B as the arrival set and we denote $f|_A \to B$ the application from A to B which associates. This function has the same rule of calculation as f, only the domain and codomain change.

Note 2.2.18

When we restrict only the domain ($B = F$), we use the notation $f/_A$.

Definition 2.2.19

Let f and g be functions, we say that f is an extension of g if g is a restriction of f.

Example 2.2.20

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$, defined as :

$$x \mapsto f(x) = x^2,$$

$$x \mapsto g(x) = x^2.$$

   That is, g is the restriction of f to $\mathbb{R}^+$,

$$g = f/_{\mathbb{R}^+} \to \mathbb{R}^+.$$

   Note that g is increasing and bijective, but f is not.

2. Let $g : \mathbb{R}^* \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, defined as :

$$x \mapsto g(x) = \frac{\sin x}{x},$$

$$x \mapsto f(x) = \begin{cases} \frac{\sin x}{x}, \text{ if } x \neq 0 \\ 1, \text{ if } x = 0 \end{cases}$$

   The function f is an extension of g,

$$g = f/_{\mathbb{R}^*}.$$

   Moreover, we can show that f is continuous on $\mathbb{R}$; and we say that f is the extension by continuity of g.