Chapter 01

simple Integrals and multiple Integrals

I. Indefinite and definite integrals - Riemann integrals
II. Double integrals and Area
III. Triple integrals and Volume

I. The integral and Riemann integral

1. Anti-derivative function and Indefinite Integrals

Definition 01: Let $f: I \to \mathbb{R}$ be a function defined on an interval I.

A function F is called an anti-derivative of f on I if F is differentiable and $F'(x) = f(x)$ On I.

Proposition 01 : If F is an anti-derivative of the function f then : $G = F + C \quad (C \in \mathbb{R})$ is an anti-derivative of the function f.

Anti-derivative of the usual functions:

Function f	Anti-derivatives $F + C$
$x^n \quad (n \in \mathbb{R} - \{-1\})$	$\frac{1}{n+1} x^{n+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$
$\frac{1}{1+x^2}$	$Arctan(x) + C$
e^x	$e^x + C$
$\sin(ax + b)$	$\frac{-1}{a}\cos(ax + b) + C$
$\cos(ax + b)$	$\sin(ax + b) + C$
$\frac{1}{\sqrt{1-x^2}}$	$Arcsin(x) + C$
$\sinh(x)$	$\cosh(x) + C$
$\cosh(x)$	$\sinh(x) + C$
$\frac{1}{\cos^2(x)}$	$\tan(x) + C$

Definition 02: The set of all anti-derivatives of a function f on I, is called the indefinite integral of f.

Written $\int f(x)dx = F(x) + C$ with $F' = f$, and C is an arbitrary constant .

2. Definite Integrals

Definition 03 : We say a definite integral of a function f on $[a \quad b]$ the real number denoted by

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Theorem: Every continuous function f on an interval I is integrable.

Properties: let f, g tows continuous functions on $[a \ \ b]$. and $\alpha, \beta \in \mathbb{R}$.

1) $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx \quad , \quad \int_a^a f(x)dx = 0$

3) $\int_{-a}^a f(x)dx = \begin{cases} 0 & \text{if } f \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f \text{ is an even function} \end{cases}$

4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx , \quad a < c < b$

5) If $f \geq 0$ on $[a \quad b]$ Then: $\int_a^b f(x)dx \geq 0$

6) If $f \leq g$ on $[a \quad b]$ Then : $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

7) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

8) The average value of f on $[a \ \ b]$ is : $\mu = \frac{1}{b-a}\int_a^b f(x)dx$

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3. Methods of intégration :

Integration by parts: Let the functions u and v have continuous derivatives on $[a \quad b]$.

$$\text{Then : } \int_a^b u\,v' = [u\,v]_a^b - \int_a^b u'v$$

Examples :

1) $\int_0^{\pi} (1-2x)\sin(x)\,dx = ?$ Put : $\begin{cases} u = 1-2x \\ v' = \sin(x) \end{cases} \Longrightarrow \begin{cases} u' = -2 \\ v = -\cos(x) \end{cases}$

$$\int_0^{\pi} x\sin(x)\,dx = [-(1-2x)cos(x)]_0^{\pi} - 2\int_0^{\pi} \cos(x)dx = (1-2\pi)+1-2[\sin(x)]_0^{\pi} = 2(1-\pi)$$

2) $\int_0^{1} \arcsin(x)dx = ?$ Put : $\begin{cases} u = \arcsin(x) \\ v' = 1 \end{cases} \Longrightarrow \begin{cases} u' = \dfrac{1}{\sqrt{1-x^2}} \\ v = x \end{cases}$

$$\int_0^{1} \arcsin(x)\,dx = [xarcsin(x)]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - 1.$$

<u>Integration by substitution</u> : Let a continuous function f on $I = [a \;\; b]$.

and $u : J \subseteq \mathbb{R} \to I$, $u(t) = x$ having a continuous derivative and the inverse

Function. Then $\int_a^b f(x)dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f\big(u(t)\big)u'(t)dt$

Examples :

1) $\int_0^1 \sqrt{1-x^2}dx = ?$ Put : $x = u(t) = \sin(t) \Rightarrow \begin{cases} u'(t) = \cos(t) \\ x = 0 \to t = 0, \;\; x = 1 \to t = \dfrac{\pi}{2} \end{cases}$

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\;\cos(t)dt = \int_0^{\frac{\pi}{2}} \cos^2(\mathrm{t})\,dt$$

$$\cos(2t) = \cos^2(\mathrm{t}) - \sin^2(t) = 2\cos^2(\mathrm{t}) - 1 \quad\Longrightarrow\quad \cos^2(\mathrm{t}) = \frac{1}{2}(1+\cos(2\mathrm{t}))$$

$$\int_0^{\frac{\pi}{2}} \cos^2(\mathrm{t})\,dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\mathrm{t}))dt = \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

2) $\int \frac{1}{(x+3)\sqrt{2+x}}dx = ?$ put: $t=\sqrt{2+x} \Rightarrow x = t^2-2 \Rightarrow dx = 2tdt$

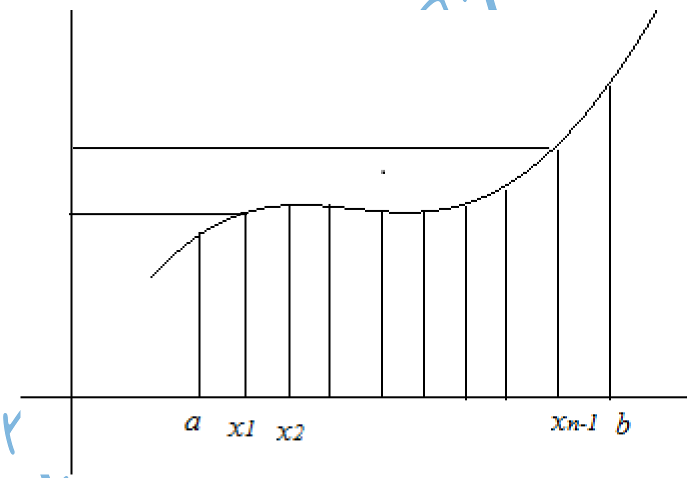
$$\int \frac{1}{(x+3)\sqrt{2+x}}dx = \int \frac{2}{t^2+1}dt = 2\arctan(t) + C = 2\arctan\left(\sqrt{2+x}\right) + C.$$

3) $\int_1^e \frac{ln(x)}{x(4+ln^2(x))}dx = ?$ Put : $t = ln^2(x) \Rightarrow \begin{cases} dt = 2\frac{ln(x)}{x}dx \\ x = 1 \to t = 0 \, , x = e \to t = 1 \end{cases}$

$$\int_1^e \frac{ln(x)}{x(4+ln^2(x))}dx = 2\int_0^1 \frac{dt}{4+t} = [2ln(4+t)]_0^1 = 2(\ln(5) - \ln(4))$$

4. Intégrale de Riemann :

Let f a continuous function on the interval $[a \;\; b]$.



Definition 01 : A partition of $[a \;\; b]$ with subintervals is determined by the set of endpoints $\{x_0, x_1, \dots\dots\dots.x_n\}$ / $x_0 = a < x_1 < \dots\dots\dots. < x_n = b$

Definition 02 : We say the lenght of the subinterval $[x_i \;\; x_{i+1}]$, $\Delta_k = \max_{1\le k\le n}(x_k - x_{k-1})$

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Remark 01 : In this part we consider a unioforme (regular) partition.

$$\Delta_k = (x_k - x_{k-1}) = \frac{b-a}{n} \quad \text{and} \quad x_k = a + \frac{b-a}{n}\, k$$

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Definition 03 :

$$\text{The sums } \mathrm{S_n} = \sum_{k=1}^{n} f(x_k)\,\Delta_k = \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{b-a}{n}\, k\right)$$

$$\text{and} \quad s_n = \sum_{k=0}^{n-1} f(x_k)\,\Delta_k = \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a + \frac{b-a}{n}\, k\right)$$

Are called The upper and lower Sum of Riemann of f on $[a \;\; b]$.

Definition 04 : that f is said to be Riemann integrable on $[a \;\; b]$ if

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \mathrm{s}_n \ \text{ Exists and finite. denoted by } \int_a^b f(x)dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$$

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Theorem 01 : If f is a continuous function on the intervalle $[a \;\; b]$, Then, the Riemann sums

$$(S_n) \text{ and } (s_n) \text{ converge to } \int_a^b f(x)dx \, .$$

Examples :

$$1) \;\; J = \int_1^4 (x^2 - 3x + 1)dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{b-a}{n}\, k\right)$$

$$\frac{b-a}{n} = \frac{3}{n} \quad , \quad a + \frac{b-a}{n}\, k = 1 + \frac{3}{n} k$$

$$J = \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n}\left(1 + \frac{3}{n}\, k\right)^2 - 3\left(1 + \frac{3}{n}\, k\right) + 1 = \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n}\left(-1 - \frac{3}{n}k + \frac{9}{n^2}k^2\right)$$

$$J = \lim_{n\to+\infty}\left[\frac{3}{n}\sum_{k=1}^{n}(-1) - \frac{9}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2\right] = \lim_{n\to+\infty}\left(-3 - \frac{9}{n^2}\frac{n(n+1)}{2} + \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right)$$

$$J = -3 - \frac{9}{2} + 9 = \frac{3}{2}.$$

2) Determine the following les sums:

$$S_1 = \sum_{k=1}^{k=+\infty} \frac{k}{n^2+k^2} \quad , \quad S_2 = \frac{1}{n}\sum_{k=0}^{k=+\infty} \frac{k}{\sqrt{3n^2+k^2}}$$

$$S_1 = \sum_{k=1}^{n=+\infty} \frac{k}{n^2+k^2} = \lim_{n\to+\infty} S_n \quad \text{Avec } S_n = \sum_{k=1}^{k=n} \frac{k}{n^2+k^2}$$

$$S_n = \sum_{k=1}^{k=n} \frac{k}{n^2+k^2} = \sum_{k=1}^{k=n} \frac{k}{n^2\left(1+\frac{k^2}{n^2}\right)} = \frac{1}{n}\sum_{k=1}^{k=n} \frac{k/n}{1+\left(\frac{k}{n}\right)^2} = \left(\frac{b-a}{n}\right)\sum_{k=1}^{k=n} f\left(a+\frac{b-a}{n}k\right) = \frac{b-a}{n}\sum_{k=1}^{k=n} f\left(a+\frac{k}{n}\right)$$

$$\begin{cases} b-a=1 \\ f\left(a+\frac{k}{n}\right) = \frac{k/n}{1+\left(\frac{k}{n}\right)^2} \end{cases} \quad \text{On prend } \begin{cases} [a \quad b] = [0 \quad 1] \\ f(x) = \frac{x}{1+x^2} \end{cases}$$

f is continuous on the interval $[0 \quad 1]$, then the sum (S_n) converge, and

$$S_1 = \int_0^1 f(x)dx = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\left(\ln(1+x^2)\right)_0^1 = \frac{\ln(2)}{2}.$$

$$S_2 = \frac{1}{n}\sum_{k=0}^{n=+\infty} \frac{k}{\sqrt{3n^2+k^2}} = \lim_{n\to+\infty} S_n \quad \text{Avec } S_n = \frac{1}{n}\sum_{k=0}^{k=n-1} \frac{k}{\sqrt{3n^2+k^2}}$$

$$S_n = \frac{1}{n}\sum_{k=0}^{k=n-1} \frac{k}{\sqrt{3n^2+k^2}} = \frac{1}{n}\sum_{k=1}^{k=n} \frac{k/n}{\sqrt{3+\left(\frac{k}{n}\right)^2}} = \frac{b-a}{n}\sum_{k=1}^{k=n} f\left(a+\frac{k}{n}\right)$$

$$\begin{cases} b-a=1 \\ f\left(a+\frac{k}{n}\right) = \frac{k/n}{\sqrt{3+\left(\frac{k}{n}\right)^2}} \end{cases} \quad \text{On prend } \begin{cases} [a \quad b] = [0 \quad 1] \\ f(x) = \frac{x}{\sqrt{3+x^2}} \end{cases}$$

f is continuous on the interval $[0 \quad 1]$, then the sum (S_n) converge, and

$$S_2 = \int_0^1 f(x)dx = \int_0^1 \frac{x}{\sqrt{3+x^2}}dx = \left(\sqrt{3+x^2}\right)_0^1 = 2-\sqrt{3}.$$

Integrating rational functions :

A rational function $\frac{P(x)}{Q(x)}$ can be represented in the form of a sum of a polynomial and a proper rational function which can be splitted into partial fractions.

$$\frac{1}{(x-\alpha)^n} \text{ (first type) , et } \frac{ax+b}{[(x-\alpha)^2+\beta^2]^n} \text{ (second type) } a, b, \alpha, \beta \in \mathbb{R} \text{ et } n \in \mathbb{N}^*$$

Examples :

1) $$\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4} = x + \frac{a}{x-2} + \frac{b}{x+2}$$

Applying the method of comparing coefficients, we obtaint $a = b = 2$

$$\frac{x^3}{x^2-4} = x + \frac{2}{x-2} + \frac{2}{x+2}$$ (The denominator has real simple roots).

2) $$\frac{1}{x^2+x+1} = \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$ (The denominator has no real roots).

- Integration of $\frac{1}{(x-\alpha)^n}$, $n \in \mathbb{N}^*$

Put : $$I_n = \int \frac{1}{(x-\alpha)^n}\, dx = \begin{cases} ln|x-a| + c & n = 1 \\ \frac{-1}{n-1}\frac{1}{(x-a)^{n-1}} + c & n > 1 \end{cases}$$

- Integration of $\frac{ax+b}{[(x-\alpha)^2+\beta^2]^n}$, $n \in \mathbb{N}^*$

Put : $I_n = \int \frac{ax+b}{[(x-\alpha)^2+\beta^2]^n}\, dx$ Let's put the change of variable $x = \beta t + \alpha$

$$I_n = \int \frac{a(\beta t+\alpha)+b}{(\beta^2 t^2+\beta^2)^n}\,\beta\, dt = \frac{1}{\beta^{2n-1}}\int \frac{a\beta t + a\alpha + b}{(1+t^2)^n}\, dt = \frac{a}{\beta^{2n-2}}\int \frac{t}{(1+t^2)^n}\, dt + \frac{a\alpha+b}{\beta^{2n-1}}\int \frac{dt}{(1+t^2)^n}$$

Put : $J_n = \int \frac{dt}{(1+t^2)^n}$ and $L_n = \int \frac{t}{(1+t^2)^n}\, dt$

- Integration of L_n

$$L_n = \int \frac{t}{(1+t^2)^n}\, dt \qquad \text{Posons } \ u = 1+t^2 \ L_n = \frac{1}{2}\int \frac{1}{u^n}\, du$$

$$L_n = \begin{cases} \frac{1}{2}\, ln|u| + c = \frac{1}{2}\, ln|1+t^2| + c & n = 1 \\ \frac{-1}{2(n-1)}\,\frac{1}{u^{n-1}} + c = \frac{-1}{2(n-1)}\,\frac{1}{(1+t^2)^{n-1}} + c & n > 1 \end{cases}$$

- Integration of J_n

$$J_n = \int \frac{dt}{(1+t^2)^n} \quad \text{By parts put:} \quad \begin{cases} u = \frac{1}{(1+t^2)^n} \\ v' = 1 \end{cases} \Rightarrow \begin{cases} u' = \frac{-2nt}{(1+t^2)^{n+1}} \\ v = t \end{cases}$$

$$J_n = \frac{t}{(1+t^2)^n} + 2n\int \frac{t^2}{(1+t^2)^{n+1}}\, dt = \frac{t}{(1+t^2)^n} + 2n\int \frac{t^2+1}{(1+t^2)^{n+1}}\, dt - 2n\int \frac{1}{(1+t^2)^{n+1}}\, dt$$

$$J_n = \frac{t}{(1+t^2)^n} + 2n\int \frac{1}{(1+t^2)^n}\, dt - 2n\int \frac{1}{(1+t^2)^{n+1}}\, dt = \frac{t}{(1+t^2)^n} + 2n\, J_n - 2n\, J_{n+1}$$

we have arrived at the recurrence relation :
$$\begin{cases} J_{n+1} = \frac{2n-1}{2n}\, J_n + \frac{1}{2n}\frac{t}{(1+t^2)^n} & n \geq 1 \\ J_1 = \arctan(t) + c \end{cases}$$

Examples :

1) $\int \frac{x^3 - x + 2}{x^2 - 1} dx = ?$ $\quad \frac{x^3 - x + 2}{x^2 - 1} = x + \frac{2}{x^2-1} = x + \frac{a}{x-1} + \frac{b}{x+1} = x + \frac{1}{x-1} - \frac{1}{x+1}$

$$\int \frac{x^3 - x + 2}{x^2 - 1} dx = \int \left(x + \frac{1}{x-1} - \frac{1}{x+1}\right) dx = \frac{x^2}{2} + ln|x-1| - ln|1+x| + C = \frac{x^2}{2} + ln\left|\frac{x-1}{x+1}\right| + C$$

2) $I = \int \frac{x+1}{x^2 - 3x + 3} dx = ?$ The denominator has no real roots.

We have : $x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[1 + \frac{\left(x - \frac{3}{2}\right)^2}{\frac{3}{4}}\right] = \frac{3}{4}\left[1 + \left(\frac{2x-3}{\sqrt{3}}\right)^2\right]$

Put : $\frac{2x-3}{\sqrt{3}} = t$, $\quad \left(x = \frac{\sqrt{3}\, t + 3}{2}\right) \Rightarrow \begin{cases} dx = \frac{\sqrt{3}}{2} dt \\ x + 1 = \frac{\sqrt{3}\, t + 5}{2} \end{cases} \Rightarrow I = \frac{\sqrt{3}}{3}\int \frac{\sqrt{3}\, t + 5}{1 + t^2} dt$

$$I=\int \frac{t}{1+t^2}dt+\frac{5}{\sqrt{3}}\int \frac{dt}{1+t^2}=\frac{1}{2}ln(1+t^2)+\frac{5}{\sqrt{3}}\arctan(t)+C$$

$$I=\frac{1}{2}ln\left(1+\left(\frac{2x-3}{\sqrt{3}}\right)^2\right)+\frac{5}{\sqrt{3}}\arctan\left(\frac{2x-3}{\sqrt{3}}\right)+C$$

Integrating of the trigonometric functions :

- Integrals of the rational functions in *sin*, *cos and tan*

In this case we put the substitution : $t=tan\left(\frac{x}{2}\right)$

$t=tan\left(\frac{x}{2}\right)\ \Rightarrow x=2\arctan(t)$ and $dx=\frac{2}{1+t^2}\,dt$

$$\begin{cases}\sin(x)=\sin\left(2\frac{x}{2}\right)=2\cos\left(\frac{x}{2}\right)sin\left(\frac{x}{2}\right)=\dfrac{2\cos\left(\frac{x}{2}\right)sin\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}=\dfrac{2t}{1+t^2}\\ \cos(x)=\dfrac{1-t^2}{1+t^2}\\ \tan(x)=\dfrac{2t}{1-t^2}\end{cases}$$

Examples :

1) $\int \frac{1}{\sin(x)}\,dx=\int\frac{dt}{t}=ln|t|+c=ln\left|tan\left(\frac{x}{2}\right)\right|+c$

2) $\int_0^{\frac{\pi}{3}}\frac{\tan(x)}{1+\cos(x)}\,dx$? Posons $t=tan\left(\frac{x}{2}\right)$, $dx=\frac{2t}{1+t^2}\,dt$, $\tan(x)=\frac{2t}{1-t^2}$, $\cos(x)=\frac{1-t^2}{1+t^2}$.

$$\int_0^{\frac{\pi}{3}}\frac{\tan(x)}{1+\cos(x)}\,dx=\int_0^{\frac{1}{\sqrt{3}}}\frac{2t}{1-t^2}=[-ln|1-t^2|]_0^{\frac{1}{\sqrt{3}}}=-ln\left|1-\frac{1}{3}\right|=ln\left(\frac{3}{2}\right)$$

3) $\int\frac{dx}{2+cos(x)}=\int\frac{1+t^2}{3+t^2}\frac{2}{1+t^2}dt=\frac{2}{3}\int\frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2}=\frac{2}{\sqrt{3}}arctan\left(\frac{t}{\sqrt{3}}\right)+C$

$$\int\frac{dx}{2+cos(x)}=\frac{2}{\sqrt{3}}arctan\left(\frac{tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)+C$$

- Integrals of the form $\int \sin^p(x)\cos^q(x)dx$

1) p even , and q odd : we put the substitution $t = \sin(x)$, $dt = \cos(x)\,dx$

Example :

$$\int \sin^2(x)\cos^3(x)dx = \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx = \int t^2(1-t^2)\,dt = \frac{t^3}{3} - \frac{t^5}{5} + c$$
$$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c$$

2) p odd, and q even : we put : $t = \cos(x)$, $dt = -\sin(x)\,dx$

Example :

$$\int \sin^3(x)\cos^2(x)dx = \int \sin(x)\left(1-\cos^2(x)\right)\cos^2(x)dx = \int (t^2-1)t^2 dt = \frac{1}{5}t^5 - \frac{1}{3}t^3 + c$$
$$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + c$$

3) p et q odds : Put : $t = \cos(x)$ or $t = \sin(x)$
4) p et q evens : In this case written $\sin(x)$ and $\cos(x)$ by applying the trigonometric identities ($\sin(2x)$ and $\cos(2x)$) .

Example

$$\int \sin^2(x)\cos^2(x)dx \ ?$$

We have $\begin{cases} \cos(2x) = \cos^2(x) - \sin^2(x) \\ 1 = \cos^2(x) + \sin^2(x) \end{cases} \Longrightarrow \begin{cases} \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \\ \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \end{cases}$

$$\sin^2(x)\cos^2(x) = \frac{1}{4}\left(1-\cos^2(2x)\right) = \frac{1}{4}\left[1 - \frac{1}{2}(1+\cos(4x))\right] = \frac{1}{8}(1-\cos(4x))$$
$$\int \sin^2(x)\cos^2(x)dx = \frac{1}{8}\int (1-\cos(4x))dx = \frac{1}{8}x - \frac{1}{32}\sin(4x) + c.$$

Integration of exponential functions :

Putting : $t = e^x$, $dt = e^x dx$ et $dx = \frac{dt}{t}$

Example :

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{t}{1+t}\,dt = \int \left(1 - \frac{1}{1+t}\right)dt = t - ln|1+t| + c = e^x - ln\,|1+e^x| + c$$

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II. Double Integrals :

1) Double integrals over a rectangle

Definition 01 : Let f be a continuous function of two variables on the domain $[a \quad b] \times [c \quad d]$

And $\{x_0, x_1 \dots, x_n\}, \{y_0, y_1 \dots, y_n\}$ a regular partition of $[a \quad b]$ and $[c \quad d]$ resp.

We have : $\Delta_x = x_i - x_{i-1} = \dfrac{b-a}{n}$, $\Delta_y = y_j - y_{j-1} = \dfrac{d-c}{n}$

With : $x_i = a + \dfrac{b-a}{n} i$, $y_j = c + \dfrac{d-c}{n} j$

The function f is said to be integrable on $[a \quad b] \times [c \quad d]$ if the following limit exists and finite .

$$\lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} (\Delta_x \Delta_y f(x_i, y_j)) = \lim_{n\to\infty}\left(\frac{(b-a)(d-c)}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(a+\frac{b-a}{n}i, c+\frac{d-c}{n}j)\right)$$

We writing : $$\int_a^b\int_c^d f(x,y)dx\,dy = \lim_{n\to\infty}\left(\frac{(b-a)(d-c)}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(a+\frac{b-a}{n}i, c+\frac{d-c}{n}j)\right)$$

Example : $I = \int_0^1\int_0^1 xe^y\,dx\,dy = ?$

$$I = \lim_{n\to\infty}\left(\frac{(b-a)(d-c)}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(a+\frac{b-a}{n}i, c+\frac{d-c}{n}j)\right) = \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\frac{i}{n},\frac{j}{n})\right)$$

$$I = \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i}{n}\,e^{\frac{j}{n}}\right) = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{i=1}^{n}(i)\sum_{j=1}^{n}\left(e^{\frac{1}{n}}\right)^j\right)$$

$$\begin{cases} \displaystyle\sum_{i=1}^{n}(i) = \frac{n(n+1)}{2} \\ \displaystyle\sum_{j=1}^{n}\left(e^{\frac{1}{n}}\right)^j = e^{\left(\frac{1}{n}\right)}\frac{e-1}{e^{\frac{1}{n}}-1} \end{cases}$$

$$I=\lim_{n\to\infty}\left(\frac{e^{\left(1+\frac{1}{n}\right)}-1}{2}\,\frac{n(n+1)}{n^2}\,\frac{\frac{1}{n}}{e^{\frac{1}{n}}-1}\right)=\frac{e-1}{2}.$$

Fubini's Theorem : Let f a continuous function over a domain $D=[a\quad b]\times[c\quad d]$.

We have : $$\iint_D f(x,y)dxdy=\int_c^d\left(\int_a^b f(x,y)dx\right)dy=\int_a^b\left(\int_c^d f(x,y)dy\right)dx$$

.

Example : $I=\int_0^1\int_0^1(2x+y)dxdy=?$

$$\begin{cases} I=\int_0^1\left(\int_0^1(2x+y)dx\right)dy=\int_0^1(1+y)dy=\frac{3}{2}\\ I=\int_0^1\left(\int_0^1(2x+y)dy\right)dx=\int_0^1\left(2x+\frac{1}{2}\right)dx=\frac{3}{2}\end{cases}$$

2) Double integral over a non-rectangular domain: Let f a continuous fonction over a Domain $D\subseteq\mathbb{R}^2$.

D represented by one of the following formes :

$$D=\{(x,y)\in\mathbb{R}^2\ ,\quad a\le x\le b\ \text{ et }\ \varphi(x)\le y\le\psi(x)\ \}$$

$$\iint_D f(x,y)dxdy=\int_a^b\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$

$$D=\{(x,y)\in\mathbb{R}^2\ ,\quad c\le y\le d\ \text{ et }\ \varphi(y)\le x\le\psi(y)\ \}$$

$$\iint_D f(x,y)dxdy=\int_c^d\left(\int_{\varphi(y)}^{\psi(y)} f(x,y)dx\right)dy$$

Examples :

1) $I = \iint_D y\,dxdy = ?\quad D = \{(x,y) \in \mathbb{R}^2\ ,\quad x \geq 0\,, y \geq 0\ \text{ et }\ x + y \leq 1\,\}$

$$\begin{cases} I = \int_0^1 \left(\int_0^{1-x} ydy\right) dx = \frac{1}{2}\int_0^1 (x^2 - 2x + 1)dx = \frac{1}{6} \\ \text{ou} \\ I = \int_0^1 y\left(\int_0^{1-y} dx\right) dy = \int_0^1 (y - y^2)dy = \frac{1}{6} \end{cases}$$

3) Propreties :

Let f, g be continuous functions over a domain $D \subseteq \mathbb{R}^2$, and α, β a real numbers.

1) $\iint_D (\alpha f(x,y) + \beta g(x,y))dxdy = \alpha \iint_D f(x,y)dxdy + \beta \iint_D g(x,y)dxdy$

2) $\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$ with $D = D_1 \cup D_2$

.

3) If $f \geq 0$ on D , then $\iint_D f(x,y)dxdy \geq 0$.

4) If $f \geq g$ on D, then $\iint_D f(x,y)dxdy \geq \iint_D g(x,y)dxdy$

5) $\left|\iint_D f(x,y)dxdy\right| \leq \iint_D |f(x,y)|dxdy$

.

6) $\iint_{[a\ \ b]\times[c\ \ d]} f(x,y)dxdy = \left(\int_a^b f_1(x)dx\right)\left(\int_c^d f_2(y)dy\right)$ with $f(x,y) = f_1(x)f_2(y)$

Example : $I = \int_0^1 \int_0^1 \frac{2x+y}{1+x^2} dxdy = ?$

$$I = \int_0^1 \int_0^1 \frac{2x+y}{1+x^2} dxdy = \int_0^1 \int_0^1 \left(\frac{2x}{1+x^2} + \frac{y}{1+x^2} \right) dxdy = 2 \int_0^1 \frac{x}{1+x^2} dx + \left(\int_0^1 \frac{dx}{1+x^2} \right) \left(\int_0^1 y dy \right)$$

$$I = \left(ln(1+x^2) \right)_0^1 + (\arctan(x))_0^1 \left(\frac{1}{2} y^2 \right)_0^1 = \ln(2) + \frac{\pi}{8}$$

.

4) Change of variables in triple integrals : Let f be a continuous function on $D \subseteq \mathbb{R}^2$.

Affine coordinates : Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, bijective . $\varphi(u,v) = (x,y)$ we have :

$$\iint_D f(x,y) dxdy = \iint_\Delta f\big(\varphi(u,v), \varphi(u,v)\big) |J| dudv \ ; \ \Delta = \varphi^{-1}(D) \text{ , and } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example : $I = \iint_D (x+y) dxdy$ where $D = \{(x,y) \in \mathbb{R}^2 \ , \ 1 \le x - y \le 2 , -1 \le x + 3y \le 1\}$

Put : $\begin{cases} u = x - y \\ v = x + 3y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4}(3u + v) \\ y = \frac{1}{4}(-u + v) \end{cases}$, $\begin{cases} J = \frac{1}{4} \\ \Delta : \ 1 \le u \le 2 , -1 \le v \le 1 \end{cases}$

$$I = \frac{1}{8} \iint_\Delta (u+v) dudv = \frac{1}{8} \iint_\Delta u dudv + \frac{1}{8} \iint_\Delta v dudv = \frac{1}{8} \left(\int_1^2 u du \right) \left(\int_{-1}^1 dv \right) + \frac{1}{8} \left(\int_1^2 du \right) \left(\int_{-1}^1 v dv \right)$$

$$I = \frac{1}{8} \left(\int_1^2 u du \right) \left(\int_{-1}^1 dv \right) + \frac{1}{8} \left(\int_1^2 du \right) \left(\int_{-1}^1 v dv \right) = \frac{3}{8}.$$

Polar coordinates

$$\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \end{cases} , \; J = r \; , \; \Delta = \varphi^{-1}(D) \quad \iint_D f(x,y)dxdy = \iint_\Delta f(rcos(\theta), rsin(\theta))rdrd\theta$$

Example : $I = \iint_D xydxdy$ where $D = \{(x,y) \in \mathbb{R}^2 \; , x > 0, y > 0 \; , x^2 + y^2 \leq 1 \}$

$$\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \end{cases} , \; J = r \; , \; \Delta : \; 0 \leq r \leq 1 \, , \quad 0 \leq \theta \leq \frac{\pi}{2}$$

.

$$I = \int_0^1 \int_0^{\frac{\pi}{2}} (r^3 \sin(\theta)\cos(\theta))drd\theta = \left(\int_0^1 r^3 dr\right)\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta\right) = \frac{1}{8}$$

.

5) Area :

Let D be a domain in $\mathbb{R}^2$. Area of D is given by : $A(D) = \iint_D dxdy$

Example : $D = \{(x,y) \in \mathbb{R}^2, \; 0 < y < 1 \text{ et } 0 < x < e^y\}$

$$A(D) = \iint_D dxdy = \int_0^1 \int_0^{e^y} dxdy = \int_0^1 e^y dy = e - 1$$

III. triple integrals :

Let f be a continuous function over a domain $D \subseteq \mathbb{R}^3$.

1) Definition 01 : We definite triple intégral of f on D by the real denoted :

$$I = \iiint_D f(x,y,z)dxdydz$$

.

Fubini's theorem : Let $D = [a \quad b] \times [c \quad d] \times [p \quad q] \subseteq \mathbb{R}^3$

$$\iiint_D f(x,y,z)dxdydz = \int_a^b \left(\int_c^d \left(\int_p^q f(x,y,z)dz\right)dy\right)dx$$

$$= \int_c^d \left(\int_p^q \left(\int_a^b f(x,y,z)dx\right)dz\right)dy$$

$$= \int_p^q \left(\int_a^b \left(\int_c^d f(x,y,z)dy\right)dx\right)dz$$

2) Change of variables in triple integrals :

cylindrical Coordinates : $\varphi : \begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \\ z = z \end{cases}$, $\Delta = \varphi(D)$, $-\pi \le \theta \le \pi$, $J = r$

$$I = \iiint_{\Delta} f(rsin(\theta), rsin(\theta), z)|J|drd\theta dz$$

Example : $\iiint_D zdxdydz \quad D = \{(x,y,z) \in \mathbb{R}^3,\ 0 \le z \le 1 \text{ et } x^2 + y^2 \le z^2\}$

$$D = \{(x,y,z) \in \mathbb{R}^3,\ 0 \le z \le 1 \text{ et } x^2 + y^2 \le z^2\}$$

Let's change to cylindrical coordinates : $\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \\ z = z \end{cases}$, $J = r$, $\begin{cases} 0 \le r \le z \le 1 \\ -\pi \le \theta \le \pi \end{cases}$

$$I = \int_0^1 \int_0^z \int_{-\pi}^{\pi} zd\theta drdz = \left(\int_0^1 z\left(\int_0^z dr\right)\right)\left(\int_{-\pi}^{\pi} d\theta\right) = 2\pi \int_0^1 (z^2)dz = \frac{2\pi}{3}$$

.

sphérical Coordinates :

$$\varphi : \begin{cases} x = rcos(\theta)\cos(t) \\ y = rsin(\theta)\cos(t) \\ z = rsin(t) \end{cases} \quad |J| = r^2\cos(t),\ 0 \le \theta \le 2\pi,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$\iiint_D f(x,y,z)dxdydz = \iiint_{\Delta} f(rcos(\theta)\cos(t), rsin(\theta)\cos(t), rsin(t))|J|drd\theta dt$$

Exemple : $\iiint_D zdxdydz \quad D = \{(x,y,z) \in \mathbb{R}^3, \quad x^2+y^2+z^2 \leq 1\}$

$$I = \iiint_D zdxdydz = \iiint_\Delta rsin(t) r^2 \cos(t)\, drd\theta dt = \left(\int_0^1 r^3 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\sin(2t)\right) dt$$

$$= \frac{1}{4}\, 2\pi\, \frac{1}{2} = \frac{\pi}{4}$$

3) volume :

Let D be a domain in $\mathbb{R}^3$. Volume of D is given by : $V(D) = \iiint_D dxdydz$

Example : The volume of a sphère of radius $R = 1$.

$V(D) = \iiint_D dxdydz \quad D : x^2+y^2+z^2 = 1$ Utilisons les coordonnées sphériques :

$$\begin{cases} x = rcos(\theta)\cos(t) \\ y = rsin(\theta)\cos(t) \\ z = rsin(t) \end{cases} \quad 0 < r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$V(D) = \iiint_\Delta r^2 \cos(t)\, drd\theta dt = \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\, dt\right) = \frac{4\pi}{3}$$

.