TD 01

Exercise 1

Consider the following four statements :

(a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0.$
(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0.$
(c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0.$
(d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 > x.$

1. Are the statements a, b, c, d true or false ? Provide their negations.
2. Let P, Q, and R be three statements. Verify by creating a truth table :
   (a) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$,
   (b) $\overline{(P \Rightarrow Q)} \Leftrightarrow P \wedge \overline{Q}$.

Exercise 2

Let f function of $\mathbb{R}$ in $\mathbb{R}$. Translate the following expressions into quantifier terms :

1. f is bounded above.
2. f is bounded.
3. f is even.
4. f never equals zero.
5. f is periodic.
6. f is increasing.
7. f is not the zero function.
8. f attains all values in $\mathbb{N}$.

Exercise 3

Let $f : \mathbb{R} \to \mathbb{R}$. What is the difference in meaning between the two proposed statements ?

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y = f(x)$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y = f(x)$.
2. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = f(x)$ and $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y = f(x)$.
3. $\forall x \in \mathbb{R}, \exists M \in \mathbb{R}, f(x) \leq M$ and $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$.

Exercise 4

Show by recurrence that :

1. $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$
2. $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is a multiple of 9.

Exercise 5

By the absurd show that :
$\forall n \in \mathbb{N}$, n^2 even $\Rightarrow n$ is even.

Exercise 6

By contrapositive, show that

1. If $(n^2 - 1)$ is not divisible by 8 then n is even.
2. $\forall \varepsilon > 0, |x| \leq \varepsilon \Rightarrow x = 0$.