Probability

Sample spaces and events

Definition: The sample space of an experiment denoted S, is the set of all possible outcomes of that experiment.

Example: Observing the gender of the next child born at the local hospital, where $S=\{M, F\}$

Events

Definition: An event is any collection (subset) of outcomes contained in the sample space S

Properties of Probability

Given an experiment and a sample space S, the objective of probability is to assign to each event A a number $P(A)$, called the probability of the event A, which will give a precise measure of the chance that A will occur.

Properties of Probability

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, \ldots, A_i, \ldots$, is an infinite collection of disjoint events, then $P(A_1 \cup A_2 \cup \cdots) = \sum_{i=1}^{+\infty} P(A_i)$.

④ $P(\phi) = 0$. ⑤ $P(A) \leq 1, \forall A \in S$.

Consider an event A, with $N(A)$ denoting the number of outcomes contained in A. Then

$$P(A) = \frac{N(A)}{N}$$, where N is the outcomes number of the experiments.

Discrete Random Variables and probability Distributions

Random variables: In any experiment, there are numerous characteristics that can be observed or measured, but in most cases an experimenter will focus on some specific aspect or aspects of a sample.

Definition: For a given sample space S of some experiment, a random variable (r.v) is ~~an~~ function whose domain is the sample space and whose range is the set of real numbers.

Example

Bernoulli random Variable

Definition: Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

Types of Random variables

We distinguished between to different types of random variables.

Definition: A discrete random variable is an random variable whose possible values either constitute a finite set or else be listed in an infinite sequence in which there is a first element, a second element, and so on ---

A random variable is continuous if both of the following apply:

1. Its set of possible values consists either of all numbers in a single interval on the number line (possible from $-\infty$ to $+\infty$) or all numbers in a disjoint union of such intervals (e.g., $[0,10] \cup [20,30]$).
2. No possible value of the variable has positive probability, that is, $P(X=c)=0$ for any possible value c.

Probability Distributions for discrete Random Variables

Example: Suppose that the probability distribution of X is as given in the following table; the first row of the table lists the possible X values and the second row gives the probability of each such value.

x	0	1	2	3	4	5	6
$P(x)$	0,05	0,10	0,15	0,25	0,20	0,15	0,10

We have: $P(X \leq 2) = P(X = 0 \text{ or } 1 \text{ or } 2)$

$$= P(0) + P(1) + P(2)$$

$$= 0{,}05 + 0{,}10 + 0{,}15 = 0{,}30$$

$$P(X \geq 3) = P(X = 3 \text{ or } 4 \text{ or } 5 \text{ or } 6)$$

$$= 1 - P(X \leq 3) = 1 - 0{,}3 = 0{,}7.$$

Definition: The probability distribution or probability mass function is defined for every number x by $p(x) = P(X = x) = P(\text{all } s \in S : X(s) = x)$

Example: Six lots of components are ready to be shipped by a certain supplier. The number of defective components in each lot is as

Lot	1	2	3	4	5	6
Number of defectives	0	2	0	1	2	0

One of these lots is to be randomly selected for shipment to a particular customer. Let X be the number of defectives in the selected lot. The three possible X values are: 0, 1, and 2

$p(0) = p(X=0) = p(\text{lot } 1 \text{ or } 3 \text{ or } 6 \text{ is sent}) = \frac{3}{6} = 0{,}5$

$p(1) = p(X=1) = p(\text{lot } 4 \text{ is sent}) = \frac{1}{6} = 0{,}167$

$p(X=2) = p(2) = p(\text{lot } 2 \text{ or } 5 \text{ is sent}) = \frac{2}{6} = 0{,}333$

X	0	1	2
p(X)	0,5	0,167	0,333

The values of X along with their probabilities collectively specify the probability distribution.

The Binomial Probability distribution

There are many experiments that conform either exactly or approximately to the following list of requirements:

(1)- The experiment consists of a sequence of n smaller experiments called trials.

(2)- Each trial can result in one of the same possible outcomes (dichotomous trials), which we generically denote by success (S) and failure (F).

(3)- The trials are independent so that the outcome on any particular trial does not influence the outcome on any other trial.

(4)- The probability of success P(S) is constant from trial; we denote this probability by P.

Definition: An experiment for which conditions 1-4 are satisfied is called a binomial experiment.

Example: The same coin is tossed successively and independently n times. We arbitrarily use S to denote the outcome H (heads) and F to denote the outcome T (tails).
Then this experiment satisfies conditions 1-4.

The Binomial Random Variable and Distribution

Definition: The binomial random variable X associated with a binomial experiment consisting of n trials is defined as

X = the number of S's among the n trials.

Suppose, for example, that n = 3. Then there are eight possible outcomes for the experiment:

SSS SSF SFS SFF FSS FSF FFS FFF

From the definition of X, X(SSF) = 2, X(SFF) = 1, and so on. Possible values for X in an n-trial experiment are 0, 1, 2, ..., n.
We will often write $X \sim Bin(n, p)$ to indicate that X is a binomial r.v based on n trials with success probability p.

Example: Consider first the case $n=4$ for which each outcome, its probability, and corresponding x values are listed in the Table -A-

For example, $P(SSFS) = P(S) \cdot P(S) \cdot P(F) \cdot P(S)$ [independent trials]

$$= p \times p \times (1-p) \times p$$

$$= p^3(1-p).$$

Outcome	x	Probability	outcome	x	Probability
SSSS	4	p^4	FSSS	3	$p^3(1-p)$
SSSF	3	$p^3(1-p)$	FSSF	2	$p^2(1-p)^2$
SSFS	3	$p^3(1-p)$	FSFS	2	$p^2(1-p)^2$
SSFF	2	$p^2(1-p)^2$	FSFF	1	$p(1-p)^3$
SFSS	3	$p^3(1-p)$	FFSS	2	$p^2(1-p)^2$
SFSF	2	$p^2(1-p)^2$	FFSF	1	$p(1-p)^3$
SFFS	2	$p^2(1-p)^2$	FFFS	1	$p(1-p)^3$
SFFF	1	$p(1-p)^3$	FFFF	0	0

We wish $b(x,4,p)$ (probability mass function) for $x = 0, 1, 2, 3$ and 4.

For $b(3,4,p)$, we have: (From the table -A-)

$$b(3,4,p) = P(FSSS) + P(SFSS) + P(SSFS) + P(SSSF)$$

$$= 4p^3(1-p)$$

$$b(3;4;3) = \left\{ \begin{array}{c} \text{number of outcomes} \\ \text{with } x=3 \end{array} \right\} \times \left\{ \begin{array}{c} \text{Probability of} \\ \text{any particular outcome} \\ \text{with } x=3 \end{array} \right\}$$

Exercice: Calculate: (From the table -A-)

$b(2;4,p)$, $b(4;4,p)$, $b(1;4,p)$.

Theorem

$$b(x;n,p)=\begin{cases} C_n^x\, p^x (1-p)^{n-x} & , x=0,1,2,\dots,n \\ 0 & \text{otherwise.}\end{cases}$$

Example: Each of six randomly selected cola drinkers is given a glass containing cola S and one containing cola F. The glasses are identical in appearance except for a code on the bottom to identify the cola. Suppose there is actually no tendency among cola drinkers to prefer one cola to the other, then $P = 0.5$, so with X = the number among the six who prefers S $\Rightarrow X \sim B(6;0.5)$

Thus

$$P(X=3) = C_6^3 (0.5)^3 (0.5)^3 = \frac{6!}{3!\,3!} (0.5)^3 (0.5)^3$$

$$= \frac{6\times5\times4\times\cancel{3!}}{\cancel{3!}\,(3\times2)} (0.5)^3 \times (0.5)^3$$

$$= 20\,(0.5)^6 = \boxed{0.313}$$

The probability that at last three prefers is $P(X \leq 3) = \sum_{x=3}^{6} b(x; 6, 0{,}5)$

$$= \sum_{x=3}^{6} C_6^x (0{,}5)^x (0{,}5)^{6-x}$$

$$= C_6^3 (0{,}5)^3 (0{,}5)^3 + C_6^4 (0{,}5)^4 (0{,}5)^2 + C_6^5 (0{,}5)^5 (0{,}5) + C_6^6 (0{,}5)^6$$

where: $C_n^k = \frac{n!}{k!(n-k)!}$

$n! = n(n-1)(n-2) \times \cdots \times 2 \times 1.$