*Series of exercises 3 (questions marked * left to the students)*

Exercise 01

Determine the domain of definition for the function f in each of the following cases:

1) $f(x)=\frac{x^2}{x^2+1}$, 2) $f(x)=\sqrt{3x-x^3}$, 3) $f(x)=\sqrt{|x-1|}$, 4)* $f(x)=\sqrt{\frac{x}{6-x}}$

5) $f(x)=\sqrt{-x}+\frac{1}{\sqrt{2+x}}$, 6)* $f(x)=\sqrt{1-x}+2\sqrt{x-1}+\sqrt{x^2+1}$, 7) $f(x)=\frac{x^2+1}{E(x)}$

8) $f(x)=\frac{1}{\ln(1-x)}+\sqrt{2+x}$, 9)* $f(x)=\ln(-\ln(x^2-5x+5))$, 10) $f(x)=\frac{\sqrt{x}}{\sin \pi x}$.

Exercise 02 Using the definition, prove that:

1) $\lim_{x\to 3}\frac{x^2-1}{x^2+1}=\frac{4}{5}$, 2) $\lim_{x\to\infty}\frac{x^2-1}{x^2+1}=1$, 3) $\lim_{x\to 1}\frac{x+2}{(x-1)^2}=+\infty$, 4) $\lim_{x\to+\infty}\frac{x^2+x+1}{-x+1}=-\infty$.

5)* $\lim_{x\to-1}\frac{x+3}{x+2}=2$, 6)* $\lim_{x\to\infty}\frac{2x^2+x+1}{x^2-3x}=2$, 7)* $\lim_{\substack{> \\ x\to 1}}\frac{2x^2-x-2}{x^2-x}=-\infty$* , 8)* $\lim_{x\to-\infty}\frac{2x^2+x+1}{1-3x}=+\infty$.

Exercise 03 Calculate the following limits:

1) $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$, 2)* $\lim_{x\to 1}\left(\frac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}\right)$, 3) $\lim_{x\to+\infty} x\left(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x\right)$

4)* $\lim_{x\to-\infty}\left(\sqrt{x^2-2x}-\sqrt{x^2-1}\right)$, 5) $\lim_{x\to\mp\infty} x\ln\frac{x}{x+2}$, 6)* $\lim_{x\to\mp\infty} x\ln\frac{x^2+x}{x^2+2x+3}$

7) $\lim_{x\to\mp\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2}$, 8)* $\lim_{x\to+\infty}\left(\frac{x+3}{x-2}\right)^{2x+1}$, 9) $\lim_{x\to\pm\infty}\frac{x}{\mathrm{E}(x)+1}$, 10)* $\lim_{x\to\pm\infty} x\mathrm{E}(x)$ 11) $\lim_{x\to+\infty}\frac{x+3}{x^2-2}\sin x$, 12)* $\lim_{x\to+\infty}\frac{x+\cos(x^2+2x)}{x^2-2}$.

Remark: For questions 5,6,7 and 8, use the limit: $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$.

Exercise 04

Prove that the function f does not have a limit at x_0 in each of the following cases:

1) $x_0=0$ and $f(x)=\sin\frac{1}{x}$, 2)* $x_0=\infty$ and $f(x)=\cos x$,

3)* $x_0=\infty$ and $f(x)=x-E(x)$, 4) $x_0=0$, $f=v\circ u$, $u(x)=x\cos\frac{1}{x}$ and

$v(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$.

Exercise 05

Can the function f be continuously extended at x_0 in each of the following cases:

1) $f(x)=\frac{x^3+5x+6}{x^5+1}$ *and*$x_0=-1$, 2)* $f(x)=\frac{x^2-x}{\sin(x-1)}$ *and* $x_0=1$,

3) $f(x)=1-x\sin\frac{1}{x}$ *and* $x_0=0$.

Exercise 06

Determine a and b such that the function f is continuous in its defined domain in each of the following cases.

1) $f(x)=\begin{cases} x & si\ |x|\le 1 \\ x^2+ax+b & si\ |x|>1 \end{cases}$, 2)* $f(x)=\begin{cases} (x-1)^3 & si\ x\le 0 \\ ax+b & si\ 0<x<1. \\ \sqrt{x} & si\ x>1 \end{cases}$

Exercise 07 Apply the Mean value theorem to prove the following:

1) $\forall x\in]0;1[: 1+x<e^x<\frac{1}{1-x}$, 2)*$\forall x\ge 0 :\ 2\le\sqrt{4+x^2}\le 2+\frac{x^2}{2}$

3) $na^{n-1}(b-a)<b^n-a^n<nb^{n-1}(b-a)$ where $0<a<b$ and $n\ge 2$ be an integer.

Exercise 08 Calculate the following limits using L'Hôpital's rule:

1) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$, 2) $\lim\limits_{x\to\frac{\pi}{3}}\frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}$, 3) $\lim\limits_{x\to+\infty}x\left(e^{\frac{2x+1}{x^2}}-1\right)$, 4)* $\lim\limits_{x\to 0}\frac{\tan x-x}{x-\sin x}$

5) $\lim\limits_{x\to+\infty}\frac{\ln(2x^3-3x^2+x-1)}{2x-1}$, 6) $\lim\limits_{x\to+\infty}x\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{x}\right)\right]$, 7)* $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\pi cotan\,\pi x\right)$

8) $\lim\limits_{x\to 0}\frac{3\tan 4x-12\tan x}{3\sin 4x-12\sin x}$, 9) $\lim\limits_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$, 10)* $\lim\limits_{x\to 0}\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)$.

Exercise 09 Let the function f be defined on $\mathbb{R}$ as follows: $f(x)=\begin{cases} \frac{-x^2}{x+2} & si\ x\ge 0 \\ \ln\frac{2x^2+1}{x^2+1} & si\ x<0 \end{cases}$.

1) Examine the continuity of f over $\mathbb{R}$.

2) Is the function f differentiable at 0?

3) Express $f'(x)$ in terms of x.

4) Prove that f has an reciprocal function f^{-1} This involves defining its definition domain and expressig $f^{-1}(x)$ in terms of x.

Exercise 10 *******Let the function f be defined on $\mathbb{R}$ as follows $f(x)=\begin{cases} \frac{\sin ax}{x} & si\ x<0 \\ e^{bx}-x & si\ x\ge 0 \end{cases}$

where a and b are real numbers.

a) Determine a such that f is continuous at 0.

b) Determine the value of b so that f is differentiable at 0.

Exercise 11 Prove the following:

1) $\forall x\in]-1,1[: \tan Arc\sin x=\frac{x}{\sqrt{1-x^2}}$.

2)* $\forall x \in [-1,0[\cup]0,1]: \tan Arc\cos x = \frac{\sqrt{1-x^2}}{x}$.

3) $\forall x \in \mathbb{R}: \cos Arc\tan x = \sin Arc\, co\tan x = \frac{1}{\sqrt{1+x^2}}$.

4)* $\forall x \in \mathbb{R}: \sin Arc\tan x = \cos Arc\, co\tan x = \frac{x}{\sqrt{1+x^2}}$.

5) $\forall x \geq 0: Arc\tan(x+1) - Arc\tan x = Arc\tan\frac{1}{1+x+x^2}$.

6)* $\forall x \in [-1,1]: Arc\cos x + Arc\cos(-x) = \pi$.

Exercise 12 Calculate the following limits using L'Hôpital's rule:

1) $\lim\limits_{x\to 0}\frac{x\,\text{Arc}\sin x^2}{x\cos x - \sin x}$, 2)* $\lim\limits_{x\to 0}\left(\frac{1}{xArc\tan x} - \frac{1}{x^2}\right)$, 3) $\lim\limits_{x\to 0}(e^x + x)^{\frac{1}{x}}$ 4)* $\lim\limits_{x\to+\infty}\left(\tan\frac{\pi x}{2x+1}\right)^{\frac{1}{x}}$, 5) $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{2\cos x}$, 6) $\lim\limits_{x\to 0}\frac{\cosh x - \cos x - x^2}{x^6}$

7)* $\lim\limits_{x\to+\infty}\frac{\pi - 2Arc\tan x}{\ln\frac{x-1}{x+1}}$.