Chapter 2 : Numerical sequences

1) **Definition :**

A numerical sequence is a function f with

$$f: IN \to IR$$
$$n \to f(n),$$

where $f(n)$ is the nth term in the sequence. The sequences are denoted by $(u_n), (a_n), (x_n), \dots \dots$

Example :

1) **Let (a_n) and (x_n) is a sequences given by :**

$$a_n: IN^* \to IR$$
$$n \to \frac{1}{n}.$$

$$x_n: IN \to IR$$
$$n \to 5^n.$$

2)Increasing and decreasing sequences :

A numerical sequence (a_n) is:

1) **Strictly increasing if, for all n : $a_n < a_{n+1}$.**
2) **Increasing if, for all n : $a_n \leq a_{n+1}$.**
3) **Strictly decreasing if, for all n : $a_n > a_{n+1}$.**
4) **Decreasing if, for all n : $a_n \geq a_{n+1}$.**
5) **Monotonic if it is increasing or decreasing .**
6) **Non-monotonic if it is neither increasing nor decreasing.**
7) **Fixed if, for all n : $a_n = a_{n+1}$.**

Example Recall the sequences (a_n), (b_n) and (c_n), given by $a_n = n$, $b_n = (-1)^n$ and $c_n = \frac{1}{n}$. We see that:

1. for all n, $a_n = n < n+1 = a_{n+1}$, therefore (a_n) is strictly increasing;
2. $b_1 = -1 < 1 = b_2$, $b_2 = 1 > -1 = b_3$, therefore (b_n) is neither increasing nor decreasing, i.e. non-monotonic;
3. for all n, $c_n = \frac{1}{n} > \frac{1}{n+1} = c_{n+1}$, therefore (c_n) is strictly decreasing.

<u>Proposition 01</u> :

Let (a_n) a numerical sequences given by a regressive expression :

$$a_{n+1} = f(a_n), \forall n \in IN,$$

If f is increasing, then (a_n) is monotonic.

<u>Exemple :</u> Let the numerical sequence

$$a_{n+1} = 3a_n - 2, \forall n \in IN.$$
$$a_0 = 2.$$

We have $f(a_n) = 3a_n - 2$, with

$$f' = 3 > 0 \rightarrow f \text{ is increasing},$$

Then (a_n) is monotonic such that

$$a_1 - a_0 = 4 - 2 = 2 > 0.$$

Finally (a_n) is increasing.

<u>3)Bounded sequences :</u>

A numerical sequence (a_n) is:

1) **Bounded above if, for all n, there exists U such that : $a_n \leq U$.**
   U is an upper bound for (a_n).

2) **Bounded below if, for all n, there exists U such that : $a_n \geq U$.**
   U is an lower bound for (a_n).

3) **Bounded if it is both bounded above and bounded below.**

Example

1. The sequence $\left(\frac{1}{n}\right)$ is bounded since $0 < \frac{1}{n} \leq 1$.
2. The sequence (n) is bounded below but is not bounded above because for each value C there exists a number n such that $n > C$.

3. Given the sequence a_n=(1, 2, 1, 2, 1, 2 ...), we can see that the interval [1, 2] contains every term in a_n. This sequence is therefore a bounded sequence.

<u>4)Limit of sequence :</u>

Definition : A numerical sequence (a_n) converges to a real number l if :

$$\lim_{n\to+\infty} a_n = l$$

<u>Example :</u>

Consider the sequence (a_n):

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\dots, 1+\frac{1}{n}, \dots\dots$$

The sequence (a_n) is converge and has the limit 1.

Theorem 2.3 (Algebraic Limit Theorem). *Let* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then,*

(i) $\lim_{n\to\infty} ca_n = ca$ *for all* $c \in R$

(ii) $\lim_{n\to\infty}(a_n + b_n) = a+b$

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$

(iv) $\lim_{n\to\infty}(a_n/b_n) = a/b$ *provided* $b \neq 0$

Example 2.4. If $(x_n) \to 2$, then $((2x_n - 1)/3) \to 1$.

<u>Proposition 2</u>

Let (a_n), (b_n) and (c_n) are a numerical sequences. If

$$b_n \leq a_n \leq c_n, \forall n \in IN.$$

And

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} c_n = l.$$

Then:

$$\lim_{n\to+\infty} a_n = l.$$

<u>Example</u> : Let (a_n) a numerical sequence given by

$$\forall n \in IN^*: a_n = 1 - \frac{sin(n)}{n^2}.$$

Since : $\forall n \in IN, -1 \leq sin(n) \leq 1$, **we obtain :**

$$\forall n \in IN^*: 1 - \frac{1}{n^2} \leq a_n \leq 1 + \frac{1}{n^2}.$$

We have :

$$\lim_{n\to+\infty} 1 - \frac{1}{n^2} = \lim_{n\to+\infty} 1 + \frac{1}{n^2} = 1,$$

then:

$$\lim_{n\to+\infty} a_n = 1.$$

5)<u>Divergence sequences:</u> A sequence that does not have a limit or in other words, does not converge, is said to be divergent.

<u>Example :</u>

Consider the sequence (a_n):

$$a_n: IN \to IR$$
$$n \to (-1)^n.$$

The sequence does not converge because have two limites 1 and -1.

6)<u>Adjacent sequences</u>

<u>Definition :</u> two sequences are adjacent if the first is increasing, the second is decreasing, and their difference converges to 0.

<u>Example :</u>

Consider the sequences (a_n) and (b_n):

$$a_n = 1 + \frac{1}{n^2}, \quad b_n = 1 - \frac{1}{n^2} \quad .$$

<u>7)Arithmetic sequence</u>

Definition: An **arithmetic sequence** is a sequence of the form

$$a, a+d, a+2d, a+3d, a+4d, \dots$$

The number a is the **first term**, and d is the **common difference** of the sequence. The **nth term** of an arithmetic sequence is given by

$$a_n = a + (n-1)d$$

<u>Example :</u>

Consider the sequence (a_n):

$$1, 3, 7, \dots\dots, 2n+1, \dots\dots$$

We have

$$a_{n+1} - a_n = (2n+2) + 1 - 2n - 1 = 2,$$

Then (a_n) is a arithmetic sequence with the first term $a_0 = 1$ is and the common difference 2.

Definition: For the arithmetic sequence $a_n = a + (n-1)d$, the **nth partial sum**

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots + [a + (n-1)d]$$

is given by either of the following formulas.

$$1) S_n = n\left(\frac{a_0 + a_n}{2}\right)$$

$$2) S_n = number\ of\ terms \left(\frac{the\ first\ term + the\ last\ term}{2}\right)$$

8)Geometric sequence

Definition :

A **geometric sequence** has the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

in which each term is obtained from the preceding one by multiplying by a constant, called the **common ratio** and often represented by the symbol r. Note that r can be positive, negative or zero. The terms in a geometric sequence with negative r will oscillate between positive and negative.

It is easy to see that the formula for the nth term of a geometric sequence is

$$a_n = ar^{n-1}.$$

Example :

Consider the sequence (a_n):

$$1, 5, 25, \ldots\ldots, 5^n, \ldots\ldots$$

We have

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{5} = 5.$$

Then (a_n) is a geometric sequence with the first term $a_0 = 1$ is and the common ratio 5.

Definition : **The nth partial sum of a geometic sequence is given by :**

$$S_n = the\ first\ term \left(\frac{1 - (common\ ratio)^{number\ of\ terms}}{1 - common\ ratio}\right).$$

Proposition :

The convergence of the geometric sequences depends on the value of the common ratio a:

- **If : $-1 < a < 1$, the sequence converges .**
- **If : $a > 1$, the sequence divergents .**
- **If : $a \leq -1$, the sequence divergents.**