*Series of exercises 1 (questions marked * left to the students)*

***Exercise 01*:**

Prove the following:

1) .$\forall x; y \in \mathbb{R}$: $\left||x| - |y|\right| \leq |x + y|$

2) $\forall x; y \in \mathbb{R}$: $\forall \varepsilon > 0 : xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2} y^2$ (it's called Cauchy's inequality with ε:

3).$(\forall \varepsilon > 0 : |x| < \varepsilon) \Rightarrow (x = 0)$

4).$\forall x; y \in \mathbb{R} : |x| + |y| \leq |x + y| + |x - y|$

5).$(|x + y| = |x| + |y|) \Leftrightarrow (xy \geq 0)$

6)$\forall x_1; x_2 \dots; x_n; y_1; y_2; \dots \dots; y_n \in \mathbb{R}$: $(\sum_{i=1}^{n} x_i y_i)^2 \leq (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$

<u>Exercise 02 :</u>

Prove the following

1) $\forall \varepsilon > 0$; $\exists n \in \mathbb{N}^*$: $0 < \frac{1}{n} < \varepsilon$.

2) $\forall x; y \in \mathbb{R}: (x < y) \Rightarrow \big(E(x) \leq E(y)\big)$

3) $\forall x; y \in \mathbb{R}: E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1$

4). $\forall x \in \mathbb{R}: -1 \leq E(x) + E(-x) \leq 0$

5). $Min\ (x; y) = \frac{x+y-|x-y|}{2}$; $Max(x; y) = \frac{x+y+|x-y|}{2}$

6).$\forall n \in \mathbb{N}^*; \forall x \in \mathbb{R}: E\left(\frac{E(nx)}{n}\right) = E(x)$

specify if possible sup,.inf,.max,.min, for each set of the following :

a) $A = \left\{\frac{2n+1}{n} ; n \in \mathbb{N}^*\right\}$. *c*)* $C = \left\{\frac{1}{n} + \frac{1}{m} ; n \in \mathbb{N}^*, m \in \mathbb{N}^*\right\}$.

b) $B = \left\{\frac{1}{x^2+1}; x \in \mathbb{R}\right\}$. *d*)* $D = \left\{-2 < x + \frac{1}{2x} < 2 ; x \in \mathbb{R}^*\right\}$.

<u>Exercise 03 :</u>

1) Prove that:

a) If the natural number n is not a perfect square then $\sqrt{n}$ *is irrational*

b If) $r \in \mathbb{Q}$ *and* $x \notin \mathbb{Q}$ *then* $r + x \notin \mathbb{Q}$.

c) If $r \in \mathbb{Q}^*$ *and* $x \notin \mathbb{Q}$ *then* $r + x \notin \mathbb{Q}$.

d) The number $\sqrt{15} + \sqrt{12}$ *is rational (explain)?*

2) Let the set A defined by : $A = \{1 < x < \sqrt{8} ; x \in \mathbb{Q}\}$.

.prove that A accepts a lower bound and does not accept an upper bound in Q

3)the equation $x^3 - x + 1 = 0$ *doesn't accept solution in Q.*

Exercise 04:

1) Let there E and F be two non empty and bounded set prove that :

a). $(E \subseteq F) \Rightarrow (InfF \leq InfE \leq SupE \leq SupF)$

b).$Sup(E \cup F) = Max\{SupE, SupF\}$

c).$Inf(E \cup F) = Min\{InfE, InfF\}$

d) . and $E - F = \{x - y; x \in E$ و $y \in F\}$ $-F = \{-x; x \in F\}$

2) Prove that :

a) .$Sup(E - F) = SupE - InfF$

b) .$Inf(E - F) = InfE - SupF$

c) .$Sup(-F) = -InfF$

d) .$Inf(-F) = -SupF$

3) Let $E \subset \mathbb{R}_+^\star$ *we put* $\frac{1}{E} = \left\{\frac{1}{x}; x \in E\right\}$ *Prove that* :

a) .$Inf\frac{1}{E} = \frac{1}{SupE}$

b) If* $InfE \neq 0$ *then .*$Sup\frac{1}{E} = \frac{1}{InfE}$

Exercise 05* :

:Prove that:

1) $\forall x; y; z \in \mathbb{R}_+^\star (: x + y + z = 1) \Rightarrow \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9\right)$.

2) $\forall x; y \in \mathbb{R}: \frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.

3) The number $\frac{\ln 5}{\ln 6}$ *Is irrational ?.*

4) $\forall x; y \in \mathbb{R}: E(x) + E(y) + E(x + y) \leq E(2x) + E(2y)$

5) $\forall n \in \mathbb{N}: E\left(\sqrt{n} + \sqrt{n+1}\right) = E\left(\sqrt{4n+2}\right)$.

Exercise 06

Write $\cos^5 x$ in linear form.

Exercise 07

a) Use the De Moivre's theorem to prove that: $\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$.

b) Solve the equation $16x^4 - 12x^2 + 1 = 0$ and determine the value of $\cos\frac{\pi}{5}$.

Exercise 08

The following finite sum S , are given by

$$S = 1 + \cos\theta + \cos 2\theta + ... + \cos(n-1)\theta / \theta \neq 2\pi k, k \in \mathbb{Z} \text{ and } n \in \mathbb{N}^*$$

By using the De Moivre's theorem, prove that: $S = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} + \frac{1}{2}$.

Exercise 09*

One of the roots of equation $z^7 - 1 = 0$ is denoted by ω , where $0 < \arg\omega < \frac{\pi}{3}$.

a) Find ω in the form $re^{i\theta}, r > 0, 0 < \theta < \frac{\pi}{3}$.

b) Show clearly that $1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

c) Hence, using the results from the previous parts deduce that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

Exercise 10*

Calculate: $S = \sum_{p=0}^{n-1} \frac{\sin px}{\cos^p x}$ $\left(\text{using the sum } \sum_{p=0}^{n-1} \frac{e^{ipx}}{\cos^p x}\right)$.