1 Elements of set theory and applications

Definition

A set E is any collection of objects, called elements of set E. If the number of these objects is finite, it is called the cardinal of E and is denoted $card(E)$; if E has infinitely many elements, it is said to be of infinite cardinal and is denoted $CardE = \infty$.

If an object x is an element of E, x is said to belong to E and is denoted $x \in E$. If x is not an element of E, we note $x \notin E$.

Example

$\mathbb{N}$ ($\mathbb{R}$, $\mathbb{Z}$ respectively) is the set of natural numbers (real, integer respectively).

Parts of a set

Definition

A set A is said to be included in a set B, or A is a part of set B, or A is a subset of B if any element of A is an element of B. We note $A \subset B$ and formally have :

$$A \subset B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B).$$

Definition

When A is not a part of B, we note $A \not\subset B$ and formally have :

$$A \not\subset B \Leftrightarrow \exists x((x \in A) \wedge (x \notin B)).$$

The set of all parts of a set A is denoted $P(A)$.

Example

Let $A = \{a, b, c\}$, then

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}.$$

Property:

Let A be a set, then $\varnothing \in P(A)$ and $A \in P(A)$.

Definition

Let A and B two sets, A is said to be equal to B, denoted $A = B$, if they have the same elements.

Formally we have :

$$\begin{aligned} A &= B \Leftrightarrow \forall x(x \in A \Leftrightarrow x \in B) \\ &\Leftrightarrow (A \subset B) \wedge (B \subset A). \end{aligned}$$

Operations on sets

Definition

Let A and B be two sets.

- The set of elements of A that also belong to B is called the intersection of A and B. (denoted $A \cap B$)

- The set of elements of A and those of B is called the union of A and B. (denoted $A \cup B$)

Formally, we have :

$$A \cap B = \{x; (x \in A) \wedge (x \in B)\}.$$
$$A \cup B = \{x; (x \in A) \vee (x \in B)\}.$$

Example
Let $A = \{a, b, c, 1, 3\}$, $B = \{b, c, d, 1, 0, 8\}$, alors :

$$A \cap B = \{b, c, 1\}.$$
$$A \cup B = \{a, b, c, d, 0, 1, 3, 8\}.$$

Proposition
Let A, B and C be three parts of E, we have :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

The intersection is said to be distributive with respect to the union and vice versa.

Proof
Let's fix

$$x \in A \cap (B \cup C)$$

we have

$$[x \in A \text{ and } x \in B \cup C],$$

hence

$$(x \in A \text{ and } x \in B)$$

or

$$(x \in A \text{ and } x \in C),$$

so

$$x \in (A \cap B) \cup (A \cap C),$$

hence the inclusion in one direction.

In the other direction, consider x as an element of the second term, then

$$x \in A \cap B \text{ or } x \in A \cap C.$$

In both cases, we have

$$x \in A \text{ and } x \in B \cup C,$$

what needs to be demonstrated.

The second equality can be demonstrated in the same way.

Definition
If $A \cap B = \varnothing$ we say that A and B are two disjoint sets, and if moreover $E = A \cup B$, we say that A is the complementary of B in E, or that A and B are two complementary sets in E, and we note :

$$A = C_E B \text{ or } B = C_E A \text{ or } A = E \backslash B.$$

Property:

Let E be a set and A a part of E. The complementary of A in E is the set $C_E A$ such that

$$C_E A = \{x \in E; x \notin A\}.$$

Example

Let $E = \{1, 4, a, d, \alpha, \mu, \lambda\}$ and $A = \{4, , \alpha, \mu\}$, then

$$C_E A = \{1, a, d, \lambda\}.$$

Proposition

Let E be a set and A and B two parts of E, then :

1. $A \subset B \Leftrightarrow C_E B \subset C_E A$.

2. $C_E(C_E A) = A$.

3. $C_E(A \cap B) = C_E A \cup C_E B$.

4. $C_E(A \cup B) = C_E A \cap C_E B$.

Proof 1.

$$\begin{aligned} A &\subset B \\ &\Leftrightarrow \forall x \in E, ((x \in A) \Rightarrow (x \in B)) \\ &\Leftrightarrow ((x \notin B) \Rightarrow (x \notin A)) \\ &\Leftrightarrow \forall x \in E, ((x \in C_E B) \Rightarrow (x \in C_E A)) \\ &\Leftrightarrow C_E B \subset C_E A. \end{aligned}$$

2. Let $x \in E$, then

$$\begin{aligned} x &\in C_E(C_E A) \\ &\Leftrightarrow x \notin C_E A \\ &\Leftrightarrow x \in A. \end{aligned}$$

so,

$$C_E(C_E A) = A.$$

3. Let $x \in E$, then

$$\begin{aligned} x &\in C_E(A \cap B) \\ &\Leftrightarrow x \notin A \cap B \\ &\Leftrightarrow (x \notin A) \vee (x \notin B) \\ &\Leftrightarrow (x \in C_E A) \vee (x \in C_E B) \\ &\Leftrightarrow x \in (C_E A \cup C_E B). \end{aligned}$$

so

$$C_E(A \cap B) = C_E A \cup C_E B.$$

4. Let $x \in E$, then

$$\begin{array}{ccl} x & \in & C_E(A \cup B) \\ & \Leftrightarrow & x \notin A \cup B \\ & \Leftrightarrow & (x \notin A) \wedge (x \notin B) \\ & \Leftrightarrow & (x \in C_E A) \wedge (x \in C_E B) \\ & \Leftrightarrow & x \in (C_E A \cap C_E B). \end{array}$$

so

$$C_E(A \cup B) = C_E A \cap C_E B.$$

Remark

From the first property we deduce that :

$$C_E E = \varnothing.$$

<definition/>

The product of two sets E and F, denoted $E \times F$, is the set of pairs (x, y) such that $x \in E$ and $y \in F$, i.e.

$$E \times F = \{(x, y)/x \in E \text{ et } y \in F\}.$$

We agree that

$$\forall (x, y),\ (x', y') \in A \times\ B,\ (x, y) = (x', y') \Leftrightarrow (x = x') \wedge (y = y').$$

Example

Let $A = \{1, 2\}$, $B = \{3, 4\}$, then

$$A \times\ B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Proposition

For $(A, B) \in [P(E)]^2$, $(C, D) \in [P(F)]^2$, we have the following relations

1. $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

2. $(A \times C) \cup (A \times D) = A \times (C \cup D)$.

3. $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.

Proof

Let's show the first equality, the other two are treated in the same way.

$$\begin{array}{rcl} (A \times C) \cup (B \times C) & = & \{(x, y) : (x, y) \in A \times C \text{ ou } (x, y) \in B \times C\} \\ & = & \{(x, y) : (x \in A \text{ et } y \in C) \text{ ou } (x \in B \text{ et } y \in C)\} \\ & = & \{(x, y) : (x \in A \text{ ou } x \in B) \text{ et } y \in C)\} \\ & = & (A \cup B) \times C. \end{array}$$

2 Applications and functions

Definition

An application of a set E in a set F is any correspondence f between the elements of E and those of F which to any element $x \in E$ maps a single element $y \in F$ denoted $f(x)$.

- $y = f(x)$ is called the image of x and x is an antecedent of y.
- The application f from E into F is represented by $f : E \to F$.
- E is called the starting set and F the target set of the application f.

Formally, a correspondence f between two non-empty sets is an application if and only if :

$$\forall x, x' \in E : \ ((x = x') \Rightarrow (f(x) = f(x'))) \,.$$

Example

1) f defined by :

$$\begin{array}{ccc} f & : & \mathbb{R} \to \mathbb{R} \\ x & \longmapsto & x^2 + 4 \end{array}$$

is an application.

2) f defined by :

$$\begin{array}{ccc} f & : & \mathbb{R} \to \mathbb{R} \\ x & \longmapsto & \dfrac{x}{x-1} \end{array}$$

is not an application because there is an element $x = 1$ belonging to the starting set that has no image in the target set.

Definition

1) Two applications f and g are said to be equal if:

i. They have the same starting set E and the same target set F.

ii. $\forall x \in E,\ f(x) = g(x)$.

2) The graph of an application $f : E \to F$ is the set

$$\Gamma_f = \{(x, f(x)), x \in E\} \,.$$

Composition of applications

Definition

Let $f : E \to F$ and $g : F \to G$, let $g \circ f$ be the application of E in G defined by :

$$\forall x \in E,\ gof(x) = g(f(x)).$$

This application is called the composition of applications f and g.

Example

Given the applications

$$\begin{array}{ccc} f & : & \mathbb{R} \to \mathbb{R}_+ \\ x & \longmapsto & x^2 \end{array} \quad , \quad \begin{array}{c} g : \mathbb{R}_+ \to \mathbb{R}_+ \\ x \longmapsto x^3 \end{array}$$

So,

$$g \circ f : \mathbb{R} \to \mathbb{R}_+ , \qquad g : \mathbb{R}_+ \to \mathbb{R}_+$$
$$x \longmapsto \left(x^2\right)^3 = x^6 \qquad x \longmapsto \left(x^3\right)^2 = x^6$$

It is clear that $f \circ g \neq g \circ f$.

Restriction and extension of an application

Definition

Given an application $f : E \to F$..

1. We call the restriction of f to a non-empty subset X of E, the application $g : X \to F$ such that

$$\forall x \in X,\ g(x) = f(x)$$

We note $g = f_X$.

Given a set G such that $E \subset G$, we call an extension of the application f to the set G, any application h from G into F such that f is the restriction of h to E.

Example

Given the application

$$f : \mathbb{R}_+ \to \mathbb{R}$$
$$x \longmapsto \log x$$

so,

$$g : \mathbb{R} \to \mathbb{R}_+ , \qquad h : \mathbb{R}_+ \to \mathbb{R}_+$$
$$x \to \log|x| \qquad x \to \log(2|x| - x)$$

are two different extensions of f to $\mathbb{R}$.

Images and reciprocal images

<definition/>

Let $A \subset E$ and $M \subset F$.

1. We call the image of A by f the set of images of the elements of A denoted :

$$f(A) = \{f(x), x \in A\} \subset F$$

2. The reciprocal image of M by f is the set of antecedents of the elements of M, denoted by

$$f^{-1}(M) = \{x \in E, f(x) \in M\} \subset E$$

Formally we have :

$$\forall y \in F,\ (y \in f(A) \Leftrightarrow \exists x \in A, y = f(x))$$
$$\forall x \in E,\ \left(x \in f^{-1}(M) \Leftrightarrow f(x) \in M\right).$$

Proposition

Let $f : E \to F$, A, $B \subset E$ and M, $N \subset F$, then

1. $f(A \cup B) = f(A) \cup f(B)$

2. $f(A \cap B) \subset f(A) \cap f(B)$
3. $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$
4. $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$
5. $f^{-1}(C_F M) = C_E f^{-1}(M)$.

Proof

1. Let $y \in F$, then

$$\begin{aligned} y &\in f(A \cup B) \\ &\Leftrightarrow \exists x \in A \cup B; y = f(x) \\ &\Leftrightarrow \exists x\, [(x \in A) \vee (x \in B) \wedge (y = f(x))] \\ &\Leftrightarrow [\exists x \ \ (x \in A) \wedge (y = f(x))] \wedge [\exists x \ \ (x \in B) \vee (y = f(x))] \\ &\Leftrightarrow (y \in f(A)) \vee (y \in f(B)) \\ &\Leftrightarrow y \in f(A) \cup f(B). \end{aligned}$$

which shows that

$$f(A \cup B) = f(A) \cup f(B).$$

2. Let $y \in F$, then

$$\begin{aligned} y &\in f(A \cap B) \\ &\Leftrightarrow \exists x \in A \cap B; y = f(x) \\ &\Leftrightarrow \exists x\, [(x \in A) \wedge (x \in B) \wedge (y = f(x))] \\ &\Leftrightarrow [\exists x \ \ (x \in A) \wedge (y = f(x))] \wedge [\exists x \ \ (x \in B) \wedge (y = f(x))] \\ &\Leftrightarrow (y \in f(A)) \wedge (y \in f(B)) \\ &\Leftrightarrow y \in f(A) \cap f(B). \end{aligned}$$

which shows that

$$f(A \cap B) = f(A) \cap f(B).$$

3. Let $x \in E$, then

$$\begin{aligned} x &\in f^{-1}(M \cup N) \\ &\Leftrightarrow f(x) \in M \cup N \\ &\Leftrightarrow f(x) \in M \vee f(x) \in N \\ &\Leftrightarrow \big(x \in f^{-1}(M)\big) \vee \big(x \in f^{-1}(N)\big) \\ &\Leftrightarrow x \in f^{-1}(M) \cup f^{-1}(N). \end{aligned}$$

which shows that

$$f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N).$$

4. Let $x \in E$, then

$$\begin{aligned} x &\in f^{-1}(M \cap N) \\ &\Leftrightarrow f(x) \in M \cap N \\ &\Leftrightarrow f(x) \in M \wedge f(x) \in N \\ &\Longleftrightarrow \big(x \in f^{-1}(M)\big) \wedge \big(x \in f^{-1}(N)\big) \\ &\Longleftrightarrow x \in f^{-1}(M) \cap f^{-1}(N). \end{aligned}$$

which shows that

$$f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N).$$

5. Let $x \in E$, then

$$\begin{aligned} x &\in f^{-1}\left(C_F M\right) \\ &\Leftrightarrow f(x) \in C_F M \\ &\Leftrightarrow (f(x) \in F) \wedge (f(x) \notin M) \\ &\Leftrightarrow (x \in E) \wedge \left(x \notin f^{-1}(M)\right) \\ &\Leftrightarrow x \in C_E f^{-1}(M). \end{aligned}$$

which shows that

$$f^{-1}\left(C_F M\right) = C_E f^{-1}(M).$$

Injective, surjective, bijective applications

Definition

Let $f : E \to F$ be an application

1) f is injective if and only if

$$\forall x, x' \in E, f(x) = f(x') \Rightarrow x = x'.$$

2) f is surjective if and only if

$$\forall y \in F, \exists x \in E, f(x) = y.$$

3) f is bijective $\Leftrightarrow$ f is injective and surjective if and only if

$$\forall y \in F, \exists ! x \in F; f(x) = y.$$

The reciprocal application

Proposition

An application $f : E \to F$ is bijective if and only if there exists a unique application $g : F \to E$ such that

$$fog = Id_F \text{ and } gof = Id_E.$$

We say that f is invertible and g is called the "reciprocal application" or "inverse application" of f. (denoted f^{-1})

Example

Consider the application

$$\begin{aligned} f &: \mathbb{R} - \{2\} \to F \\ x &\longmapsto \frac{x+5}{x-2} \end{aligned}$$

with F a subset of $\mathbb{R}$. Determine F so that the application f is bijective and give the inverse application of f.

To show that f is bijective is to examine the existence of solutions to the equation $y = f(x)$, for all $y \in F$.

Let $y \in F$, then

$$\begin{aligned} y &= f(x) \\ &\Leftrightarrow y = \frac{x+5}{x-2} \\ &\Leftrightarrow y(x-2) = x+5 \\ &\Leftrightarrow yx - x = 5 + 2y \\ &\Leftrightarrow x(y-1) = 5 + 2y \\ &\Leftrightarrow x = \frac{5+2y}{y-1} \text{ si } y \neq 1 \end{aligned}$$

which shows that :

$$\forall y \in \mathbb{R} - \{1\}, \exists! x = \frac{5+2y}{y-1}; \, y = f(x).$$

to show that f is bijective, it remains to be seen whether

$$x = \frac{5+2y}{y-1} \in \mathbb{R} - \{2\}?$$

We have :

$$\begin{aligned} \frac{5+2y}{y-1} &= 2 \Leftrightarrow 5 + 2y = 2(y-1) \\ &\Leftrightarrow 5 = -2 \text{ what is impossible} \end{aligned}$$

which shows that $\frac{5+2y}{y-1} \in \mathbb{R} - \{2\}$, then

$$\forall y \in \mathbb{R} - \{1\}, \exists! x = \frac{5+2y}{y-1} \in \mathbb{R} - \{2\}; \, y = f(x),$$

so, is bijective if $F = \mathbb{R} - \{1\}$ and the inverse of f is :

$$\begin{aligned} f^{-1} \quad &: \quad \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{2\} \\ y \quad &\rightarrow \quad \frac{5+2y}{y-1}. \end{aligned}$$

Functions

Definition

A function from E into F is any application f from a subset $D_f \subset E$ into F. $D_f$is called the "Definition set of f".

Remark

All the notions given for applications can be adapted for functions.

3 Binary relationships

Definition

A binary relationship is any assertion between two objects, which may or may not be verified. We note xRy and read "x is in relation to y".

Definition

Given a binary relation R between the elements of a non-empty set E, we say that :

1. R is Reflexive if and only if

$$\forall x \in E : (xRx)$$

2. R is Transitive if and only if

$$\forall x, y, z \in E : (xRy) \wedge (yRz) \Rightarrow (xRz).$$

3. R is symmetric if and only if

$$\forall x, y \in E : (xRy) \Rightarrow (yRx).$$

4. R is Antisymmetric if and only if

$$\forall x, y \in E : (xRy) \wedge (yRx) \Rightarrow x = y.$$

Equivalence relations

Definition

A binary relation R on a set E is said to be an equivalence relation if it is Reflexive, Symmetric and Transitive.

Definition

Let R be an equivalence relation on a set E.

- Two elements x and $y \in E$ are said to be equivalent if xRy.

- The equivalence class of an element $x \in E$ is the set :

$$\dot{x} = \overline{x} = \{y \in E; xRy\}.$$

- The set of equivalence classes of all elements of E is called the quotient set of E by the equivalence relation R. This set is denoted E/R.

Example

1) Given E a non-empty set, then

Equality is an equivalence relation in E

2) In $\mathbb{R}$ we define the relation R by :

$$\forall x, y \in \mathbb{R} : xRy \Leftrightarrow x^2 - 1 = y^2 - 1.$$

Show that R is an equivalence relation and give the quotient set $\mathbb{R}/R$.

1. R is an equivalence relation.

i) R is a Reflexive relation, because we have :

$$\forall x \in \mathbb{R}, x^2 - 1 = x^2 - 1,$$

so,

$$\forall x \in \mathbb{R},\ xRx$$

which shows that R is a Reflexive relationship.
ii) R is a Symmetric relation, because we have :

$$\begin{aligned} \forall x, y &\in \mathbb{R},\ xRy \\ &\Leftrightarrow x^2 - 1 = y^2 - 1 \\ &\Leftrightarrow y^2 - 1 = x^2 - 1 \\ &\Leftrightarrow yRx. \end{aligned}$$

which shows that R is a Symmetrical relation.
iii) R is a Transitive relation, because we have :

$$\begin{aligned} \forall x, y, z &\in \mathbb{R} : (xRy) \wedge (yRz) \\ &\Leftrightarrow \left(x^2 - 1 = y^2 - 1\right) \wedge (y^2 - 1 = z^2 - 1) \\ &\Leftrightarrow x^2 - 1 = z^2 - 1 \\ &\Leftrightarrow xRz. \end{aligned}$$

which shows that R is a Transitive relation.
From i) , ii) and iii), we deduce that R is an equivalence relation.
2. Determine the quotient set $\mathbb{R}/R$.
Let $x \in \mathbb{R}$, then :

$$\begin{aligned} \forall y &\in \mathbb{R},\ xRy \Leftrightarrow x^2 - 1 = y^2 - 1 \\ &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x - y)(x + y) = 0 \\ &\Leftrightarrow (y = x) \vee (y = -x) \end{aligned}$$

so:

$$\dot{x} = \{x, -x\},$$

as a result

$$\mathbb{R}/R = \{\{x, -x\}, x \in \mathbb{R}\}.$$

Proposition
Let R be an equivalence relation on a non-empty set E, then

$$\forall x, y \in E,\ (\dot{y} \cap \dot{x} = \varnothing) \vee (\dot{y} = \dot{x}).$$

Proof
Let $x, y \in E$, assume that

$$\dot{y} \cap \dot{x} \neq \varnothing$$

so,

$$\exists z \in \dot{y} \cap \dot{x},$$

thus

$$zRy \text{ et } zRx.$$

Let us then show that

$$\dot{y} = \dot{x}.$$

Let $u \in \dot{x}$, then

$$((uRx) \wedge (zRx)) \wedge (zRy)$$

as R is symmetric and transitive, we deduce that

$$(uRz) \wedge (zRy)$$

and from the transitivity of R we deduce that

$$uRy,$$

as a result

$$u \in \dot{y},$$

which shows that

$$\dot{x} \subset \dot{y}.$$

In the same way, we show that

$$\dot{y} \subset \dot{x},$$

which completes the proof of the property.

Remark

From this property we deduce that :

$$E/R \text{ est une partition de l'ensemble } E.$$

Order relations

Definition

A binary relation R on E is said to be an order relation if it is Reflexive, Transitive and Anti-Symmetric.

Definition

Let R be an order relation on a set E.

1. Two elements x and y of E are said to be comparable if :

$$xRy \text{ ou } yRx.$$

2. We say that R is a relation of total order, if all the elements of E are comparable in pairs. If not, we say that the relation R is a partial order relation.

Example

Let F be a set and $E = P(F)$.

Consider, on $E = P(F)$, the binary relation "$\subset$", then :

I) "$\subset$" is an order relation on E.
1. "$\subset$" is Reflexive, because for any set $A \in P(A)$, we have

$$A \subset A.$$

2. "$\subset$" is Transitive, because for all $A, B, C \in P(A)$,

$$\begin{aligned}(A &\subset B) \wedge (B \subset C) \\ &\Rightarrow \forall x \ \left((x \in A) \Rightarrow (x \in B)\right) \wedge \left((x \in B) \Rightarrow (x \in C)\right) \\ &\Rightarrow \forall x \ \left((x \in A) \Rightarrow (x \in C)\right) \\ &\Rightarrow A \subset C.\end{aligned}$$

3. "$\subset$" is anti-symmetric, because for all A, $B \in P(A)$,

$$(A \subset B) \wedge (B \subset A) \Leftrightarrow A = B.$$

From 1), 2) and 3) we deduce that "$\subset$" is an order relation on E.
II) Is the order total?
i) If $F = \varnothing$, then $E = \{\varnothing\}$ and we have : $\forall A, B \in E,\ A = B = \varnothing$, so

$$\forall A, B \in E,\ A \subset B$$

which shows that the order is Total.
ii) If $F = \{a\}$, then $E = \{\varnothing, \{a\}\}$, so for all A and B in E we have

$$((A = \varnothing) \vee (A = \{a\})) \wedge ((B = \varnothing) \vee (B = \{a\}))$$

so

$$\forall A, B \in E,\ \left((A \subset B) \vee (B \subset A)\right)$$

which shows that the order is Total.
iii) If F contains at least two distinct elements a and b, then

$$\exists A = \{a\},\ B = \{b\} \in E; (A \not\subseteq B) \wedge (B \not\subseteq A)$$

so A and B are not comparable, hence "$\subset$" is a partial order relation in E.
Remark
In the literature, order relations are often noted as $\preceq$.