

Numerical Series

Definition: Given a sequence $(u_n)_{n \geq 0}$, we use the notation $\sum_{i=1}^n u_i$ to denote the finite sum $u_1 + u_2 + u_3 + \dots + u_n = S_n$.

We associate with the sequence (u_n) a sequence (S_n) of partial sums.

The pair $((u_n), (S_n))$ is said to be a numerical series or a series, we denoted the series by the symbolic expression $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

If (S_n) converges to a real number S , then the series is said to be converge and we write

$$\sum_{n=1}^{\infty} u_n = S.$$

We say that the number S is the sum of the series, where $S = \lim_{n \rightarrow \infty} S_n$.

If the sequence (S_n) diverges, then the series is said to diverge.

Theorem (Necessary condition)

If $\sum u_n$ converges, then $\lim_{n \rightarrow \infty} u_n = 0$.

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Example ① Let the series: $\sum e^n$

We have $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} e^n = +\infty \neq 0$

then the series $\sum_{n \geq 0} e^n$ diverges

② $\sum_{n \geq 1} \frac{1}{n}$, $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$

but the series $\sum_{n \geq 1} \frac{1}{n}$ diverges (as we will see later)

Theorem: Let $\sum u_n$ and $\sum v_n$ be two convergent series.

a) for every $\alpha \in \mathbb{R}$: the series $\sum \alpha u_n$ and $\sum (u_n \pm v_n)$ are convergent, and we have

$$\sum \alpha u_n = \alpha \sum u_n \text{ and } \sum (u_n \pm v_n) = \sum u_n \pm \sum v_n$$

Series of nonnegative terms

We mean that $u_n \geq 0$, for every n .

Theorem (Comparison test)

1) If $|u_n| \leq v_n$, $\forall n \geq n_0$ ($n_0 \in \mathbb{N}^*$), and if $\sum (v_n)$ converges, then $\sum u_n$ converges.

2) If $v_n \geq u_n \geq 0$, $\forall n \geq n_0$ ($n_0 \in \mathbb{N}^*$), and if $\sum (u_n)$ diverges, then $\sum v_n$ diverges.

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- ③ Suppose $u_n, v_n > 0$ for all n large enough and $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = k \in [0, +\infty]$
- then if $\sum v_n$ converges and $k < +\infty$, then $\sum u_n$ converges.
- if $\sum v_n$ diverges and $k > 0$, then $\sum u_n$ diverges.

Theorem (Geometric series)

- If $|x| < 1$, then $1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$.
- If $|x| \geq 1$, the series diverges.

Tests of convergence

Theorem (Root Criterion of Cauchy)

Given $\sum u_n$, if there exists the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{|u_n|} = l. \text{ Then:}$$

- if $l < 1$, $\sum u_n$ converges.
- if $l > 1$, $\sum u_n$ diverges.
- if $l = 1$, the test gives no information.

Example: Let the series $\textcircled{1} \sum e^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{e^n} = \lim_{n \rightarrow \infty} (e^n)^{1/n} = \lim_{n \rightarrow \infty} e = e > 1$$

Then $\sum e^n$ diverges

② $\sum \frac{1}{2^n}$, $u_n = \frac{1}{2^n} > 0$

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2} < 1$$

Then $\sum \frac{1}{2^n}$ converges

Theorem (D'Alembert criterion)

If $u_n > 0$, and if the limit

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$$

If $l < 1$, then $\sum u_n$ converges

If $l > 1$, then $\sum u_n$ diverges

If $l = 1$, there is no information.

Riemann Series

Theorem: The Riemann series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

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Examples: ① $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{(1/2)}}$

The series ~~diverges~~ ^{diverges} ~~converges~~ (Riemann series with $\alpha = 1/2 < 1$).

② $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{(3/2)}}$

The series ^{converges} (Riemann series with $\alpha = \frac{3}{2} > 1$).

Multi variable functions

Functions with two variables

Let $D \subseteq \mathbb{R}^2$ and let $f: D \rightarrow \mathbb{R}$ be a function.

Fix $(x_0, y_0) \in D$.

We define the partial derivative of f with respect to x at (x_0, y_0) to be the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \text{ provided}$$

this limit exists. It is denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0)$$

We define the partial derivative of f with respect to y at (x_0, y_0) to be the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

provided this limit exists, it is denoted

$$\text{by } \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0).$$

Examples: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$(x, y) \mapsto f(x, y) = x^2 + y^2 + 10$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 2y_0, \quad \forall (x_0, y_0) \in \mathbb{R}^2$$

Let $f(x, y) = x e^y$, then $\frac{\partial f}{\partial x}(x_0, y_0) = e^{y_0}$

$$\frac{\partial f}{\partial y}(x_0, y_0) = x_0 e^{y_0}.$$

The second partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} f_x(x_0, y_0) = f_{xx}(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (f_y)(x_0, y_0) = f_{yy}(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) (x_0, y_0) = \frac{\partial}{\partial x} (f_y)(x_0, y_0) = f_{yx}(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) (x_0, y_0) = \frac{\partial}{\partial y} (f_x)(x_0, y_0) = f_{xy}(x_0, y_0)$$

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Multi-variable functions (m, 2)

$$\text{Let } f: \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x, y)$$

$$(x_1, x_2, x_3, \dots, x_n) \mapsto f(x_1, x_2, x_3, \dots, x_n)$$

$$\frac{\partial f}{\partial x_i}(x_{10}, x_{20}, \dots, x_{n0}) = \lim_{h \rightarrow 0} \frac{f(x_{10}, x_{20}, \dots, x_{i0} + h, \dots, x_{n0}) - f(x_{10}, x_{20}, \dots, x_{i0}, \dots, x_{n0})}{h}$$

Examples:

$$(1) f(x_1, x_2, x_3) = x_1^2 + x_2^3 + x_3^4$$

$$\frac{\partial f}{\partial x_1}(x_{10}, y_{10}, z_{10}) = 2x_{10} \Rightarrow \frac{\partial^2 f}{\partial x_1^2}(x_{10}, y_{10}, z_{10}) = 2$$

$$\frac{\partial f}{\partial x_2}(x_{10}, y_{10}, z_{10}) = 3y_{10}^2 \Rightarrow \frac{\partial^2 f}{\partial x_2^2}(x_{10}, y_{10}, z_{10}) = 6y_{10}$$

$$\frac{\partial f}{\partial x_3}(x_{10}, y_{10}, z_{10}) = 4z_{10}^3 \Rightarrow \frac{\partial^2 f}{\partial x_3^2}(x_{10}, y_{10}, z_{10}) = 12z_{10}^2$$

$$(2) f(x, y) = \cos x \cdot \sin y$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\sin x_0 \cdot \sin y_0 \Rightarrow \frac{\partial^2 f}{\partial x^2}(x_0, y_0) = -\cos x_0 \cdot \sin y_0$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \cos x_0 \cdot \cos y_0 \Rightarrow \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = -\cos x_0 \cdot \sin y_0$$