Chapter 2

Improper integrals

2.1 Integrals of functions defined over an unbounded interval

Our goal in this chapter is to compute integrals over unbounded intervals (ranging up to $-\infty$, 0, or $+\infty$), or integrals on a bounded interval of functions having an infinite limit at some point of the integration interval, a type of integral called an improper integral. Example: $\int_{-1}^{1} \frac{1}{x} dx = ?$

Why is the answer unclear?

Definition 2.1 *The integral defined by $\int_a^b f(x)dx$ is called an improper or generalized integral if:*

1. *$f(x)$ tends toward infinity at one or more points in $[a, b]$ (point of infinite discontinuity).*

2. *At least one of the integration bounds is infinite ($\pm\infty$).*

Example 2.1 *(a) $\int_1^{+\infty} \frac{dt}{t^2+1}$ is improper at $+\infty$.*

(b) $\int_0^1 \frac{dt}{\sqrt{1-t}}$ is improper at 1.

(c) $\int_0^2 \frac{1}{x} dx$ is improper at 0.

(d) $\int_7^{\infty} \frac{1}{x-5} dx$ has an infinite bound.

Definition 2.2 (Singular Point) *A point x_0 is called a singular point for a function f if it is not bounded at that point, i.e.,* $\lim_{x\to x_0} f(x) = \pm\infty$.

Definition 2.3 *Two integrals are said to be of the same nature if they are either both convergent or both divergent.*

2.1.1 Improper Integrals of Functions Defined on an Unbounded Interval

Let $f : [a, b] \to \mathbb{R}$ be a Riemann-integrable function on $[a, b]$, where at least one of the bounds of the interval $[a, b]$ is infinite.

Definition 2.4 *Let f be a Riemann-integrable function on $[a, +\infty[$. We say that the improper integral $\int_0^{+\infty} f(t)dt$ converges if the limit $\lim_{x\to+\infty} \int_0^x f(t)dt$ is finite. In this case, we define:*

$$\int_0^{+\infty} f(t)dt = \lim_{x\to+\infty} \int_0^x f(t)dt$$

In the contrary case, if:

$$\lim_{x\to+\infty} \int_0^x f(t)dt = \pm\infty$$

we say that the integral diverges.

Remark 2.1 *For $\int_{-\infty}^b f(t)dt$, the definition can be adapted in an obvious way.*

Example 2.2

1. *Let $f(t) = \frac{1}{t}$. It is defined and continuous on $[1, +\infty[$. Indeed:*

$$\lim_{x\to\infty} \int_1^x \frac{1}{t}dt = \lim_{x\to\infty} (\ln x - \ln 1) = +\infty$$

Thus, the generalized or improper integral $\int_1^{+\infty} \frac{1}{t}dt$ diverges.

2. *Let $f(t) = \frac{1}{(t-1)^2}$. It is defined and continuous on $[2, +\infty[$.*

$$\lim_{x\to\infty} \int_2^x \frac{1}{(t-1)^2}dt = \lim_{x\to\infty} \left(\frac{-1}{x-1} + 1\right) = 1$$

Thus, the generalized integral $\int_2^{+\infty} \frac{1}{(t-1)^2}dt$ converges.

3. *The integral* $\int_0^{+\infty} \frac{1}{1+t} dt$ *diverges:*

$$\int_0^n \frac{1}{1+t} dt = \lim_{n\to\infty} \ln(1+n) = +\infty$$

Let $f(t) = \frac{1}{(t-1)^2}$. *It is defined and continuous on* $]-\infty, 0]$.

$$\lim_{x\to\infty} \int_x^0 \frac{1}{(t-1)^2}\, dt = \lim_{x\to\infty} \left[\frac{-1}{t-1}\right]_x^0 = \lim_{x\to\infty} \left(1 + \frac{1}{t-1}\right) = 1.$$

Thus, the integral converges.

2.2 Improper integrals of functions defined on a bounded interval:

Let f *be a function defined on* $[a, b]$.

Definition: *Let* $f : [a, b[\to \mathbb{R}$ *be integrable on* $[a, b[$. *We say that the improper integral converges if*

$$\lim_{x\to b} \int_a^x f(t)\, dt$$

is finite. In this case, we write:

$$\int_a^b f(t)\, dt = \lim_{x\to b} \int_a^x f(t)\, dt.$$

In the contrary case, we say that the improper integral $\int_a^b f(t)\, dt$ *diverges.*

Remark 2.2 *For a function* f *defined and integrable on* $]a, b]$, *the definition adapts accordingly.*

Example 2.3

1. *Let* $f(t) = \frac{1}{t}$, *defined and continuous on* $]0, 1]$. *Indeed,*

$$\lim_{x\to 0} \int_x^1 \frac{1}{t}\, dt = \lim_{x\to 0} (-\ln x) = +\infty.$$

Thus, $\int_0^1 \frac{1}{t}\, dt$ *diverges.*

2. *Let* $f(t) = \frac{1}{\sqrt{-t}}$, *a function defined on* $[-1, 0[$ *and integrable in the sense of Riemann on* $[-1, 0[$. *Indeed,*

$$\lim_{x\to 0} \int_{-1}^x \frac{1}{\sqrt{-t}}\, dt = \lim_{x\to 0} (-2\sqrt{-x} + 2) = 2.$$

Thus, the improper integral converges on $[-1, 0[$.

Proposition 2.1 *Let f be a function defined and integrable on $[a,b]$*

Let $c \in]a,b[$ be a singular point, i.e., the function f is defined on $[a,b] \setminus \{c\}$. We say that:

$$\int_a^b f(t)\,dt$$

converges if both improper integrals,

$$\lim_{x\to c^-}\int_a^x f(t)\,dt \quad \textit{and} \quad \lim_{x\to c^+}\int_x^b f(t)\,dt,$$

are finite. In this case, we define:

$$\int_a^b f(t)\,dt = \lim_{x\to c^-}\int_a^x f(t)\,dt + \lim_{x\to c^+}\int_x^b f(t)\,dt.$$

Otherwise, we say the integral diverges.

[Chasles' Relation]

Let $-\infty \le a < b$ (resp. $a < b \le +\infty$) and let $f : [a,b[\to \mathbb{R}$ be continuous on $[a,b[$ (resp. $]a,b]$). Suppose the improper integral $\int_a^b f(t)\,dt$ converges. Let $c \in]a,b[$. Then:

1. *$\int_a^c f(t)\,dt$ converges.*

2. *Moreover, we have:*
$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

2.2.1 Positivity of the Improper Integral:

Proposition 2.2 *Let $f, g : [0,+\infty[\to \mathbb{R}$ be continuous functions. If both functions have convergent improper integrals and $f \le g$, then:*

$$\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx.$$

In particular, for a non-negative function f, if $f \ge 0$, then:

$$\int_0^{+\infty} f(x)\,dx \ge 0.$$

Corollary 2.1 (Symmetry of the Improper Integral) *Let $a \in \mathbb{R} \cup \{+\infty\}$, and let $f : [-a,a] \to \mathbb{R}$ be continuous on $]-a,a[$.*

1.

If f is even, then:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

If f is odd, then:

$$\int_{-a}^{a} f(x)\,dx = 0.$$

2.2.2 Convergence Criteria

1. Improper Bounded Integral

Theorem 2.1 *Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$. Let $f : [a, b[\to \mathbb{R}$ be a piecewise continuous function with $f \geq 0$. The improper integral $\int_a^b f(t)dt$ is convergent if and only if:*

$$\exists M \in \mathbb{R}_+, \quad \forall x \in [a, b[, \quad F(x) = \int_a^x f(t)dt \leqslant M.$$

Example 2.4 *Let's study the behavior of the integral I with:*

$$I = \int_0^2 \cos^2\left(\frac{1}{t}\right) dt.$$

Thus, the integral $\int_0^2 \cos^2\left(\frac{1}{t}\right) dt$ converges.

2. Riemann Integrals

A Riemann integral is:

$$\int_1^{+\infty} \frac{1}{t^2} dt.$$

In this case, the antiderivative is explicit:

$$\int_1^{+\infty} \frac{1}{t^\alpha} dt = \begin{cases} \lim_{x\to\infty} \left[\frac{1}{-\alpha+1} t^{-\alpha+1}\right]_1^x, & \text{if } \alpha \neq 1, \\ \lim_{x\to+\infty} [\ln t]_1^x, & \text{if } \alpha = 1. \end{cases}$$

We can immediately deduce the nature of Riemann integrals: if $\alpha > 1$, then $\int_1^{+\infty} \frac{1}{t^\alpha} dt$ converges; if $\alpha \leqslant 1$, then $\int_1^{+\infty} \frac{1}{t^\alpha} dt$ diverges.

Theorem 2.2 (Comparison Criterion) *Let $a < b \leqslant +\infty$ (or $-\infty < a < b$). Let f and $g : [a, b[\to \mathbb{R}$ be two continuous functions on $[a, b[$ (or $]a, b]$). Suppose:*

$$\forall x \in [a, b[, \quad 0 \leqslant f(x) \leqslant g(x).$$

Then:

- *If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.*
- *If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.*

Furthermore, in the case where both converge, we have:

$$0 \leqslant \int_a^b f(x)dx \leqslant \int_a^b g(x)dx.$$

Remark 2.3 *The comparison criterion is not applicable to functions that change sign.*

Example 2.5 *We aim to determine the convergence of:*

$$I = \int_0^{+\infty} \frac{\sin^2(t)}{1+t^2}dt.$$

The function $f(t) = \frac{\sin^2(t)}{1+t^2}$ is continuous, hence integrable in the Riemann sense over $[0, +\infty[$.

We have a potential issue with convergence at $t \to +\infty$. However, since the function is positive, we can apply the comparison criterion.

We know that:

$$0 \leq \frac{\sin^2(t)}{1+t^2} \leq \frac{1}{1+t^2} \quad \forall t \in [0, +\infty[.$$

Since the integral $\int_0^{+\infty} \frac{1}{1+t^2}dt$ converges, due to the fact that its antiderivative is $\arctan(t)$ *and* $\arctan(t) \to \frac{\pi}{2}$ *as $t \to +\infty$, we conclude that:*

$$\int_0^{+\infty} \frac{\sin^2(t)}{1+t^2}dt$$

also converges by the comparison test.

3. Absolutely Convergent Integral

Let $f : [a, b[\to \mathbb{R}$ be a function that is integrable on $[a, b[$. We say that the improper integral $\int_a^b f(t)dt$ converges absolutely if and only if:

$$\int_a^b |f(t)|dt \quad \text{converges.}$$

Proposition 2.3 *If $\int_a^b |f(t)|dt$ converges, then $\int_a^b f(t)dt$ also converges and we have the inequality:*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

Example 2.6 *Consider the integral:*

$$I = \int_1^{+\infty} \frac{\sin(t)}{t^2} dt.$$

We define the function $f(t) = \frac{\sin(t)}{t^2}$. This function is continuous on $[1, +\infty[$, and for all $t \in [1, +\infty[$, we have:

$$|f(t)| = \left|\frac{\sin(t)}{t^2}\right| \leq \frac{1}{t^2}.$$

Since the integral $\int_1^{+\infty} \frac{1}{t^2} dt$ converges (this is a standard Riemann integral), we conclude that:

$$\int_1^{+\infty} \frac{\sin(t)}{t^2} dt$$

also converges.