Chapter 02 : generalized integrals

1. Definitions and Properties
2. Tests of Convergence
3. Absolute convergence and semi-convergente

1) Definitions and Propreties

Definition 01:

Consider the function $f\ :\ I \subseteq \mathbb{R} \to \mathbb{R}$

We say that the function f is locally integrable if it is integrable over any closed and bounded sub-interval of I .

Examples : 1) Any continuous function on $I \subseteq \mathbb{R}$ is locally integrable.

2) The step functions are locally integrable.

3) All monotonic functions are locally integrable.

Definition 02 :

Let $f\ :\ I = [a\quad b[\ \to \mathbb{R}$ be localement integrable. Put : $F(x) = \int_a^x f(t)dt$

If $\lim_{x\to b} F(x) = l$ (finite), We say the integral $\int_a^b f(t)dt$ is convergent.

The limit l is called a generalized (an improper) integral of f on $[a\quad b[$. And is denoted by : $\int_a^b f(t)dt\ = l$.

Otherwise : $\left(\lim_{x\to b} F(x) = \pm\infty\right)$, The integral $\int_a^b f(t)dt$ is said to diverge .

Examples :

1) $\int_1^{+\infty} \frac{1}{(t+1)^2}dt$? $\quad F(x) = \int_1^x \frac{1}{(t+1)^2}dt = \left[\frac{-1}{1+t}\right]_1^x = \frac{-1}{1+x} + \frac{1}{2}$

$\lim_{x\to+\infty} F(x) = \frac{1}{2}$. Then $\int_1^{+\infty} \frac{1}{(t+1)^2}dt$ converges and $\int_1^{+\infty} \frac{1}{(t+1)^2}dt = \frac{1}{2}$.

2) $\int_0^{+\infty} t\,e^{-t} = ?\quad F(x) = \int_0^x t\,e^{-t}dt\quad$ By parts : $\begin{cases} u = t \\ v' = e^{-t} \end{cases} \Longrightarrow \begin{cases} u' = 1 \\ v = -e^{-t} \end{cases}$

$F(x) = [-t\,e^{-t} - e^{-t}]_0^x = 1 - xe^{-x} - e^{-x}$

$\lim_{x\to+\infty} F(x) = 1$ Alors $\int_0^{+\infty} t\,e^{-t}$ converges, and $\int_0^{+\infty} t\,e^{-t} = 1$.

3) $\int_0^1 \frac{dt}{1-t} = ?$ $F(x) = \int_0^x \frac{dt}{1-t} = -ln|1-x|$

$\lim_{x\to 1} F(x) = +\infty$ Then $\int_0^1 \frac{dt}{1-t}$ is divergent .

Remark : In the case $I =]a \ \ b]$ we put : $F(x) = \int_x^b f(t)dt$

If $\lim_{x\to a} F(x) = l$ (finite), then $\int_a^b f(t)dt$ is convergent .

If $\lim_{x\to a} F(x) = \pm\infty$ or the limit don't exists , then we say $\int_a^b f(t)dt$ diverges .

Examples :

1) $\int_0^1 \ln(t)\, dt = ?$ $F(x) = \int_x^1 \ln(t)\, dt$ By parts : $F(x) = -1 - x\, ln(x) + x$

$\lim_{x\to 0} F(x) = -1$ Then $\int_0^1 \ln(t)\, dt$ is convergent, and $\int_0^1 \ln(t)\, dt = -1.$

2) $\int_{-\infty}^0 \frac{\mathrm{dt}}{\mathrm{t}^2 + 2\mathrm{t} + 2} dt = ?$ $F(x) = \int_x^0 \frac{\mathrm{dt}}{\mathrm{t}^2 + 2\mathrm{t} + 2} dt = \int_x^0 \frac{\mathrm{dt}}{(\mathrm{t}+1)^2 + 1} dt = \left(arctan(t+1)\right)_x^0$

$F(x) = \arctan(1) - \arctan(x+1) = \frac{\pi}{4} - \arctan(x+1)$

$\lim_{x\to -\infty} F(x) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ Then $\int_{-\infty}^0 \frac{\mathrm{dt}}{\mathrm{t}^2 + 2\mathrm{t} + 2} dt$ converges, and $\int_{-\infty}^0 \frac{\mathrm{dt}}{\mathrm{t}^2 + 2\mathrm{t} + 2} dt = \frac{3\pi}{4}$.

Definition 03 : Consider the function $f :]a \ \ b[\to \mathbb{R}$ locally integrable.

we say that intgral $\int_a^b f(t)dt$ is convergent if , for any number $c \in]a \ \ b[$

the improper integrals : $\int_a^c f(t)dt$ and $\int_a^c f(t)dt$ are both convergent.

$$\text{And} \int_a^b f(t)dt = l_1 + l_2 \quad \text{With :} \quad \lim_{x\to a}\int_x^c f(t)dt = l_1 \text{ et } \lim_{x\to b}\int_c^x f(t)dt = l_2$$

Examples :

1) $$\int_{-\infty}^{+\infty} \frac{dt}{1+t^2} = \int_{-\infty}^{c} \frac{dt}{1+t^2} + \int_{c}^{+\infty} \frac{dt}{1+t^2} \quad (\text{We take } c = 0)$$

$$\int_{-\infty}^{c} \frac{dt}{1+t^2} = \lim_{x\to-\infty}\int_x^0 \frac{dt}{1+t^2} = \lim_{x\to-\infty} -\arctan(x) = \frac{\pi}{2}$$

$$\int_{c}^{+\infty} \frac{dt}{1+t^2} = \lim_{x\to+\infty}\int_0^x \frac{dt}{1+t^2} = \lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty} \frac{dt}{1+t^2} = \pi \left(\int_{-\infty}^{+\infty} \frac{dt}{1+t^2} \quad \text{converges}\right)$$

2) $$\int_{-\infty}^{0} \frac{dt}{t(t-1)} = \int_{-\infty}^{c} \frac{dt}{t(t-1)} + \int_{c}^{0} \frac{dt}{t(t-1)} \quad (\text{We take } c = -2)$$

$$\int_{-\infty}^{-2} \frac{dt}{t(t-1)} = \lim_{x\to-\infty}\int_x^{-2} \frac{dt}{t(t-1)} = \lim_{x\to-\infty}\left[-\int_x^{-2}\frac{dt}{t} + \int_x^{-2}\frac{dt}{t-1}\right] = \lim_{x\to-\infty}\left(\ln(2) + ln\left|\frac{x-1}{x}\right|\right) = \ln(2)$$

.

$$\int_{-2}^{0} \frac{dt}{t(t-1)} = \lim_{x\to0}\int_{-2}^{x} \frac{dt}{t(t-1)} = \lim_{x\to0}\left[-\int_{-2}^{x}\frac{dt}{t} + \int_{-2}^{x}\frac{dt}{t-1}\right] = \lim_{x\to0}\left(-\ln\left(\frac{3}{2}\right) + ln\left|\frac{x}{x+1}\right|\right) = -\infty$$

$\int_{-2}^{0} \frac{dt}{t(t-1)}$ is divergent, Then so is $\int_{-\infty}^{0} \frac{dt}{t(t-1)}$.

Remarks : 1) The nature of the integral of f on $]a \;\; b[$ does not depend on choice of the real c .

2) Attention : $$\int_{-\infty}^{+\infty} f(t)dt \neq \lim_{x\to+\infty}\int_{-x}^{x} f(t)dt$$

Example : $\int_{-x}^{x} t dt = \left[\frac{1}{2}t^2\right]_{-x}^{x} = 0$ but $\int_{-\infty}^{+\infty} t dt$ is divergent .

Properties : Let $f, g : [a \quad b[\rightarrow \mathbb{R}$ et $\alpha \in \mathbb{R}$. Then :

1) If $\begin{cases} \int_a^b f(t)dt \text{ converges} \\ \int_a^b g(t)dt \text{ converges} \end{cases}$ Then $\begin{cases} \int_a^b (f(t) + g(t))dt \quad \text{converges} \\ \int_a^b \alpha f(t)dt \quad \text{converges} \end{cases}$

2) If $\int_a^b f(t)dt$ diverges (or $\int_a^b g(t)dt$ diverges) Then $\begin{cases} \int_a^b (f(t) + g(t))dt \text{ diverges} \\ \int_a^b \alpha f(t)dt \quad \text{diverges} \end{cases}$

3) If $\begin{cases} \int_a^b f(t)dt \text{ diverges} \\ \int_a^b g(t)dt \text{ diverges} \end{cases}$ We can't conclude .

Example :

Let : $\int_1^{+\infty} \frac{dt}{t}$ and $\int_1^{+\infty} \frac{dt}{t+1}$ are both divergent .

$$\int_1^{+\infty} \frac{dt}{t} + \int_1^{+\infty} \frac{dt}{t+1} = \lim_{x \to +\infty} lnx(x+1) = +\infty \text{ Then : } \int_1^{+\infty} \frac{dt}{t} + \int_1^{+\infty} \frac{dt}{t+1} \text{ diverges .}$$

$$\int_1^{+\infty} \frac{dt}{t} - \int_1^{+\infty} \frac{dt}{t+1} = \lim_{x \to +\infty} \ln(2) + ln\frac{x}{x+1} = \ln(2) \text{ Then : } \int_1^{+\infty} \frac{dt}{t} - \int_1^{+\infty} \frac{dt}{t+1} \text{ converges.}$$

Riemann's integrals : $\int_{1}^{+\infty} \frac{dt}{t^{\alpha}}$, $\int_{0}^{1} \frac{dt}{t^{\alpha}}$, and $\int_{0}^{+\infty} \frac{dt}{t^{\alpha}}$

We have : $\int_{0}^{+\infty} \frac{dt}{t^{\alpha}} = \int_{0}^{1} \frac{dt}{t^{\alpha}} + \int_{1}^{+\infty} \frac{dt}{t^{\alpha}}$

For $\alpha = 1$: $\begin{cases} \int_{0}^{1} \frac{dt}{t} = \lim\limits_{x \to 0} \int_{x}^{1} \frac{dt}{t} = \lim\limits_{x \to 0} [-ln|x|] = +\infty \\ \int_{1}^{+\infty} \frac{dt}{t} = \lim\limits_{x \to +\infty} \int_{1}^{x} \frac{dt}{t} = \lim\limits_{x \to +\infty} [ln|x|] = +\infty \end{cases}$

For $\alpha < 1$: $\int_{0}^{1} \frac{dt}{t^{\alpha}} = \lim\limits_{x \to 0} \int_{x}^{1} \frac{dt}{t^{\alpha}} = \lim\limits_{x \to 0} \int_{x}^{1} t^{-\alpha} dt = \lim\limits_{x \to 0} \frac{1}{1-\alpha} [1 - x^{1-\alpha}] = \frac{1}{1-\alpha}$

$$\int_{1}^{+\infty} \frac{dt}{t^{\alpha}} = \lim_{x \to +\infty} \int_{1}^{x} \frac{dt}{t^{\alpha}} = \lim_{x \to +\infty} \int_{1}^{x} t^{-\alpha} dt = \lim_{x \to +\infty} \frac{1}{1-\alpha} [x^{1-\alpha} - 1] = +\infty$$

For $\alpha > 1$: $\int_{0}^{1} \frac{dt}{t^{\alpha}} = \lim\limits_{x \to 0} \int_{x}^{1} \frac{dt}{t^{\alpha}} = \lim\limits_{x \to 0} \int_{x}^{1} t^{-\alpha} dt = \lim\limits_{x \to 0} \frac{1}{1-\alpha} [1 - x^{1-\alpha}] = +\infty$

$$\int_{1}^{+\infty} \frac{dt}{t^{\alpha}} = \lim_{x \to +\infty} \int_{1}^{x} \frac{dt}{t^{\alpha}} = \lim_{x \to +\infty} \int_{1}^{x} t^{-\alpha} dt = \lim_{x \to +\infty} \frac{1}{1-\alpha} [x^{1-\alpha} - 1] = \frac{1}{\alpha - 1}$$

Conclusion :

$\int_{1}^{+\infty} \frac{dt}{t^{\alpha}} \begin{cases} \text{converges} & \text{if } \alpha > 1 \\ \text{diverges} & \text{if } \alpha \leq 1 \end{cases}$, $\int_{0}^{1} \frac{dt}{t^{\alpha}} \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha \geq 1 \end{cases}$

$\int_{0}^{+\infty} \frac{dt}{t^{\alpha}}$ is divergent .

Integral of Bertrand : $\displaystyle\int_a^{+\infty} \frac{dt}{t\ln^{\beta}(t)}$, $a > 1$

$$F(x) = \int_a^{x} \frac{dt}{t\ln^{\beta}(t)}$$ By change of variable : $u = \ln(t)$

For $\beta = 1$ $\quad F(x) = \ln(\ln(x)) - \ln(\ln(a))$, $\lim_{x\to+\infty}\big(\ln(\ln(x)) - \ln(\ln(a))\big) = +\infty$.

For $\beta \neq 1$ $\quad F(x) = \frac{1}{1-\beta}\left[(\ln(t))^{1-\beta}\right]_a^x = \frac{1}{1-\beta}\left((\ln(x))^{1-\beta} - (\ln(a))^{1-\beta}\right)$

$$\lim_{x\to+\infty} F(x) = \begin{cases} +\infty & \beta < 1 \\ \dfrac{(\ln(a))^{1-\beta}}{\beta - 1} & \beta > 1 \end{cases}$$ Then : $\displaystyle\int_a^{+\infty} \frac{dt}{t\ln^{\beta}(t)}$ is $\begin{cases} \text{Convergent if } \beta > 1 \\ \text{Divergent if } \beta \leq 1 \end{cases}$

2) Convergence of integrals of positive functions :

Let $f, g : [a \quad b[\ \to \mathbb{R}_+ \quad (f,\ g \geq 0)$ Locally integrals

Theorem (01) : (Test of comparison)

If $f(t) \leq g(t)$ on $[a \quad b[$ Then $\begin{cases} \displaystyle\int_a^b g(t)dt \text{ converges} \Rightarrow \int_a^b f(t)dt \text{ converges} \\ \displaystyle\int_a^b f(t)dt \text{ diverges} \Rightarrow \int_a^b g(t)dt \text{ diverges} \end{cases}$

Examples : 1) $\displaystyle\int_1^{+\infty} e^{-t^2} dt$

We have on $[1 \quad +\infty[$: $e^{-t^2} \leq e^{-t}$ et $\displaystyle\int_1^{+\infty} e^{-t}dt = \lim_{x\to+\infty}(e^{-1} - e^{-x}) = e^{-1}$

$\displaystyle\int_1^{+\infty} e^{-t}dt$ Converges Then $\displaystyle\int_1^{+\infty} e^{-t^2}dt$ is convergent.

2) $\displaystyle\int_{1}^{+\infty} \frac{\sin^2(t)}{t^2}\,dt$ On a pour $t \in [1 \quad +\infty[$ $\displaystyle\frac{\sin^2(t)}{t^2} \leq \frac{1}{t^2}$

$\displaystyle\int_{1}^{+\infty} \frac{dt}{t^2}$ Integral of Riemann converges $(\alpha > 1)$ Then $\displaystyle\int_{1}^{+\infty} \frac{\sin^2(t)}{t^2}\,dt$ converges.

Theorem (02) : (Test of equivalence)

If $\displaystyle\lim_{x\to b} \frac{f(t)}{g(t)} = 1$ We say that fonctions f and g are equivalent.

The integrals $\displaystyle\int_{a}^{b} f(t)dt$ and $\displaystyle\int_{a}^{b} g(t)dt$ have the same nature.

Examples : 1) $\displaystyle\int_{1}^{+\infty} \frac{t+1}{t(t^2+1)}$ $\displaystyle f(t) = \frac{t+1}{t(t^2+1)}$

We have $\displaystyle\lim_{x\to+\infty} \frac{f(t)}{\frac{1}{t^2}} = \lim_{x\to+\infty} \frac{t+1}{t(t^2+1)}\, t^2 = 1$ Then f and $g(t) = \dfrac{1}{t^2}$ are equivalent in the neighborhood of $(+\infty)$.

We know that $\displaystyle\int_{1}^{+\infty} \frac{1}{t^2}\,dt$ of Riemann converges $(\alpha > 1)$ Then $\displaystyle\int_{1}^{+\infty} \frac{t+1}{t(t^2+1)}\,dt$ is convergent .

2) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt$ We have $\displaystyle\lim_{x\to 0} \frac{t}{\sin(t)} = 1$ Then $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt$ and $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\mathrm{t}}\,dt$ are the same nature.

$\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\mathrm{t}}\,dt$ Of Riemann diverges $(\alpha = 1)$ Then $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt$ is divergent.

Remark : To find the equivalence of functions use limited developments .

Examples : 1) $\int_0^{\frac{1}{2}} \frac{ln(1-t^2)}{t^2}\, dt$

We have in the neighborhood of zero: $ln(1-t^2) = -t^2 - \frac{1}{2}t^4 - \frac{1}{3}t^6 + o(t^6)$

$$\frac{ln(1-t^2)}{t^2} = -1 - \frac{1}{2}t^2 - \frac{1}{3}t^4 + o(t^4)$$

Then $\int_0^{\frac{1}{2}} \frac{ln(1-t^2)}{t^2}\, dt$ is convergent, as sum of convergent integrals .

2) $\int_0^1 \frac{dt}{e^t - \cos(t)}$

In the neighborhood of zero: $e^t = 1 + t + o(t)$ and $\cos(t) = 1 + o(t)$

$e^t - \cos(t) = t + o(t)$ (Means $e^t - \cos(t) \sim t$)

$\int_0^1 \frac{1}{t} dt$ Diverges, so is $\int_0^1 \frac{dt}{e^t - \cos(t)}$.

Theorem (04) : (Test of Abel)

Consider f, g two functions locally integrables on $[a \quad b[$.

If $\begin{cases} f \text{ is monotonic and } \lim\limits_{x\to b} f(t) = 0 \\ \text{There exists } M > 0 \text{ such that } \forall x \in [a \quad b[\ : \left| \int_a^b g(t)\, dt \right| \le M \end{cases}$

Then : $\int_a^b f(t)g(t)dt$ is convergent.

Examples : 1) $\int_{1}^{+\infty} \frac{\sin(t)}{t}\, dt$

Put : $f(t) = \frac{1}{t}$ Decreasing on $[1 \ +\infty[$ and $\lim_{x\to+\infty} f(t) = 0$

We have $\left|\int_{1}^{x} \sin(t)\, dt\right| = |\cos(1) - \cos(x)| \leq |\cos(1)| + |\cos(x)| \leq 2$

$\left|\int_{1}^{+\infty} \sin(t)\, dt\right| \leq 2$, Then the integral $\int_{1}^{+\infty} \frac{\sin(t)}{t}\, dt$ is convergent.

Theorem (03) : (Test of Cauchy)

Let f be a function locally integrable on $[a \ \ b[$.

$\int_{a}^{b} f(t)dt$ converges if and only if :

$$\forall\varepsilon > 0, \exists c \in [a \ \ b[, \forall x, x' \in [a \ \ b[: \left(x > c \text{ and } x' > c \Longrightarrow \left|\int_{x}^{x'} f(t)dt\right| < \varepsilon \right)$$

3) Absolute convergence and semi convergence :

Let f be a function locally integrable on $[a \ \ b[$.

Definition (01) :

We say that $\int_{a}^{b} f(t)dt$ is absolutely convergent iff $\int_{a}^{b} |f(t)|dt$ is convergent.

Definition (02) :

We say that $\int_{a}^{b} f(t)dt$ is semi convergent if $\int_{a}^{b} f(t)dt$ converges and $\int_{a}^{b} |f(t)|dt$ *is* not convergent.

Theorem : $\int_a^b |f(t)|dt$ converges $\Longrightarrow \int_a^b f(t)dt$ converges

Example : 1) $\int_1^{+\infty} \frac{\cos(t)}{t^2}\, dt$ We have : $\left|\frac{\cos(t)}{t^2}\right| \leq \frac{1}{t^2}$

$\int_1^{+\infty} \frac{dt}{t^2}$ is convergent (Riemann) and $\int_1^{+\infty} \left|\frac{\cos(t)}{t^2}\right| dt$ is absolutely Convergent.

Then $\int_1^{+\infty} \frac{\cos(t)}{t^2}\, dt$ is convergent.