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<u>Solution of Series</u>

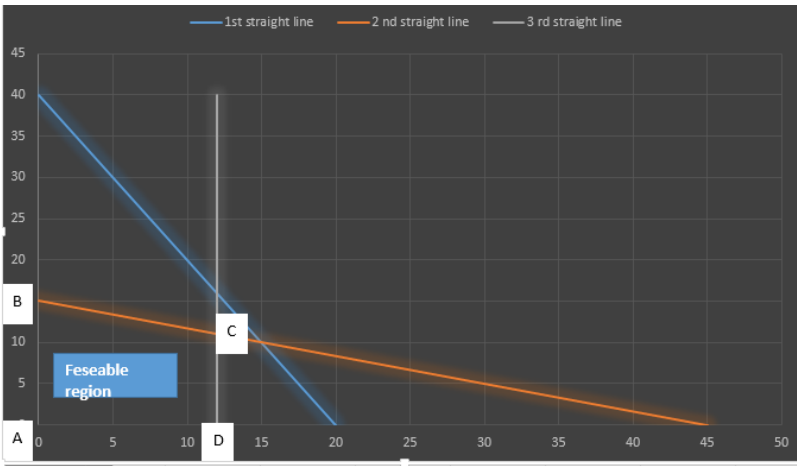
<u>Exercise 1</u>

1. Formulation of mathematical model of LP

$$\text{Max } Z=300X_1+250X_2$$
$$2X_1+X_2\leq 40$$
$$X_1+3X_2\leq 45$$
$$X_1\leq 12$$
$$X_1\geq 0,\ X_2\geq 0$$

2. Draw the graph by finding out the x_1 and x_2 intercepts

1st straight line: $2X_1+X_2= 40$			2nd straight line 2: $X_1+3X_2= 45$			3rd straight line 2: $X_1=12$		
	X_1	X_2		X_1	X_2		X_1	X_2
point 1	0	40	**point 1**	0	15	**point 1**	12	0
point 2	20	0	**point 2**	45	0	**point 2**	12	40
	The **feasible region** is the area common to the three lines and includes the points (A B C D), which allows us to determine the optimal solution, which is point C							



3. The Corners and Feasible Solution:

corner points	X_1	X_2	Z $
A	0	0	0
B	0	15	3750
C	**12**	**11**	**6350 (optimal)**
D	12	0	3600

Interpret the result. Accordingly, the highlighted result in the table above implies that 12 units of Model A and 11 units of Model B TV sets should be produced so that the total profit will be $6350.

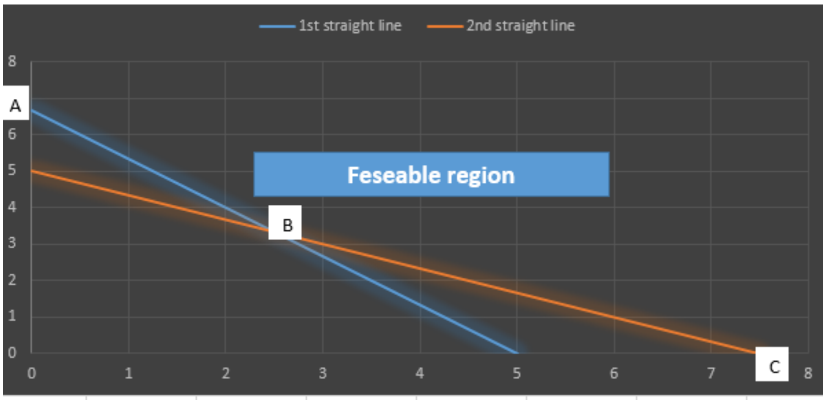
Exercise 2

1. Formulation of mathematical model of LP

$$\text{Min } Z=25X_1+30X_2$$
$$20X_1+15X_2\geq 100$$
$$2X_1+3X_2\geq 15$$
$$X_1\geq 0,\ X_2\geq 0$$

2. Draw the graph by finding out the x_1 and x_2 intercepts

1st straight line: $20X_1+15X_2= 100$			2nd straight line 2: $2X_1+3X_2= 15$		
	X_1	**X_2**		**X_1**	**X_2**
point 1	0	20/3	**point 1**	0	5
point 2	5	0	**point 2**	7.5	0
	The **feasible region** is the area common to the two lines and includes the points (A B C), which allows us to determine the optimal solution, which is point C				



3. The Corners and Feasible Solution:

corner points	X_1	X_2	Z $
A	0	20/3	200
B	**2.5**	**3.33**	**162.5(optimal)**
C	7.5	0	187.5

Since our objective is to minimize cost, the minimum amount (162.5) will be selected.

$X_1 = 2.5$, $X_2 = 3.33$ and Min Z= 162.5

Exercise 3

1. Converting the linear program to the standard form:

$$\text{Max : } Z=1X_1+X_2+4X_3+2X_4+1X_5$$
$$4X_1+X_2+1.5X_3+2.5X_4+0X_5+1S_1+0S_2+0S_3=150$$
$$2X_1+3X_2+1X_3+2X_4+7X_5+0S_1+1S_2+0S_3=180$$
$$0X_1+2X_2+2X_3+0X_4+2X_5+0S_1+0S_2+1S_3=120$$
$$X_1, X_2, X_3, X_4, X_5, S_1, S_2, S_3\geq 0$$

2. Initial Simplex Table

C_j		1	1	4	2	1	0	0	0	B_i	B_i/a_{ij}
C_b	**X_j**	**X_1**	**X_2**	**X_3**	**X_4**	**X_5**	**S_1**	**S_2**	**S_3**	**(solution)**	
0	**S_1**	4	1	1.5	2.5	0	1	0	0	150	300
0	**S_2**	2	3	1	2	7	0	1	0	180	180
0	**S_3**	0	2	2	0	2	0	0	1	120	60
Z= $\sum_{j=1}^{n} C_jX_j$		**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**Z=0**	
C_j-Z_j		**1**	**1**	**4**	**2**	**1**	**0**	**0**	**0**		

Outgoing row

Incoming column

Pivot element

-It is noted that the values of the last line include positive values, which indicates that the solution is not optimal and can be improved.

The incoming variable is: **X_3** and the outgoing variable **is S_3**

Second Simplex Table

C_j		1	1	4	2	1	0	0	0	B_i	B_i/a_{ij}
C_b	**X_j**	**X_1**	**X_2**	**X_3**	**X_4**	**X_5**	**S_1**	**S_2**	**S_3**	**(solution)**	
0	**S_1**	4	0.5-	0	5/2	-3/2	1	0	-1/4	60	24
0	**S_2**	2	2	0	2	6	0	1	-1/2	120	60
4	**X_3**	0	1	1	0	1	0	0	1/2	60	/
Z= $\sum_{j=1}^{n} C_jX_j$		**0**	**4**	**4**	**0**	**4**	**0**	**0**	**2**	**Z=240**	
C_j-Z_j		**1**	**-3**	**0**	**2**	**-3**	**0**	**0**	**-2**		

Outgoing row

Incoming column

Pivot element

-It is noted that the values of the last line include one positive value, which indicates that the solution is not optimal and can be improved.

The incoming variable is: **X_4** and the outgoing variable **is S_1**

Optimal Simplex Table

C_j		1	1	4	2	1	0	0	0	B_i
C_b	**X_j**	**X_1**	**X_2**	**X_3**	**X_4**	**X_5**	**S_1**	**S_2**	**S_3**	**(solution)**
2	**X_4**	8/5	-1/5	0	1	-3/5	2/5	0	-1/10	24
0	**S_2**	-6/5	12/5	0	0	36/5	-4/5	1	1/10	72
4	**X_3**	0	1	1	0	1	0	0	1/2	60
Z= $\sum_{j=1}^{n} C_jX_j$		**16/5**	**18/5**	**4**	**2**	**14/5**	**4/5**	**0**	**7/5**	**Z=288**
C_j-Z_j		**-11/5**	**-13/5**	**0**	**0**	**-9/5**	**-4/5**	**0**	**-7/5**	

It is clear from the values of the last line that some values are negative and some are equal to zero and therefore the optimal solution has been reached and the third basic table is the optimal table.

<u>Exercise 4</u>

1. Converting the linear program to the standard form:

Min: $Z=X_1+2X_2+0S_1+0S_2-MA_1$

$X_1-X_2-1S_1+0S_2+1A_1=3$

$2X_1+X_2+0S_1+1S_2+0A_1=10$

$X_1\geq 0, X_2\geq 0, S_1\geq 0, S_2\geq 0, A_1\geq 0$

2. Initial Simplex Table

C_j		1	2	0	0	-M	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2	A_1	(solution)	
-M	A_1	1	-1	-1	0	1	3	3
0	S_2	2	1	0	1	0	10	5
$Z=\sum_{j=1}^{n} C_j X_j$		-M	M	M	0	-M	Z=-3M	
C_j-Z_j		1+M	2-M	-M	0	0		

Outgoing row

Incoming column

Pivot element

-It is noted that the values of the last line include positive values, which indicates that the solution is not optimal and can be improved.

The incoming variable is: X_1 and the outgoing variable **is** A_1

Second Simplex Table

C_j		1	2	0	0	-M	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2	A_1	(solution)	
1	X_1	1	-1	-1	0	1	3	/
0	S_2	0	3	2	1	-2	4	4/3
$Z=\sum_{j=1}^{n} C_j X_j$		1	-1	-1	0	1	Z=3	
C_j-Z_j		0	3	1	0	-M+1		

Outgoing row

Incoming column

Pivot element

-It is noted that the values of the last line include positive values, which indicates that the solution is not optimal and can be improved.

The incoming variable is: X_2 and the outgoing variable **is** S_2

Optimal Simplex Table

C_j		1	2	0	0	-M	B_i	
C_b	X_j	X_1	X_2	S_1	S_2	A_1	(solution)	
1	X_1	1	0	-1/3	1/3	1/3	13/3	
2	X_2	0	1	2/3	1/3	-2/3	4/3	
$Z=\sum_{j=1}^{n} C_j X_j$		1	2	1	1	-1	Z=7	
C_j-Z_j		0	0	-1	1	-M+1		

Since none of the net-evaluation is positive, the optimum solution is attained and is given by Max. **Z=7 at x_1= 13/3 and x_2 =4/3.**

Exercise 5

The original primal can be written in the standard primal form as:

Max. $Z^* = -3x1 + 2x2 - 4x3$

Subject to the constraints:

$$\begin{cases} -3x1 - 5x2 - 4x3 \le -7 \\ -6\,x1 - x2 - 3x3 \le -4 \\ 7x1 - 2x2 - x3 \le 10 \\ -x1 + 2x2 - 5x3 \le -3 \\ -\,4\,x1 - 7\,x2 + 2x3 \le -2 \\ x1, x2, x3 \ge 0 \end{cases}$$

Dual primal becomes

$$\begin{cases} \text{Min. } Z^* = -7y1 - 4y2 + 10y3 - 3y4 - 2y5 \\ \text{Subject to the constraints} \\ -3y1 - 6y2 + 7y3 - y4 - 4y5 \ge -3 \\ -5y1 - y2 - 2y3 + 2y4 - 7y5 \ge 2 \\ 4y1 - 3y2 - 3y3 - 5y4 + 2y5 \ge -4 \\ y1, y2, y3, y4, y5 \ge 0 \end{cases}$$

Exercise 6:

Since the variable $x3$ is unrestricted in sign, the given LPP can be transform into standard primal form by substituting $x3 = x3\,' - x3\,''$ where $x3\,' \ge 0$, $x3\,'' \ge 0$

Standard primal becomes:

$$\begin{cases} \text{Max. } Z^* = -2x1 - 3x2 - 4(x3\,' - x3\,'') \\ \text{Subject to the constrain} \\ -2x1 - 3x2 - 5(x3\,' - x3\,'') \le -2 \\ 3x1 + x2 + 7(x3\,' - x3\,'') \le 3 \\ -3x1 - x2 - 7(x3\,' - x3\,'') \le -3 \\ x1 + 4x2 + 6(x3\,' - x3\,'') \le 5 \\ x1, x2\,, x3\,'\,, x3\,'' \ge 0 \end{cases}$$

Required dual is

Min. $Z^* = -2y1 + 3(y2\,' - y2\,'') + 5y3$

Subject to the constraints

$$\begin{cases} -2y1 + 5(y2\,' - y2\,'') + y3 \ge -2 \\ -3y1 + (y2\,' - y2\,'') + 4y3 \ge -3 \\ -5y1 + 7(y2\,' - y2\,'') + 6y3 \ge -4 \\ 5y1 - 7(y2\,' - y2\,'') - 6y3 \ge 4 \\ y1, y2\,', y2\,'', y3 \ge 0 \end{cases}$$

Again, we write

Min. $Z^* = -2y1 + 3y2 + 5y3$

Subject to the constraints

$-2y1 + 5y2 + y3 \geq -2$

$-3y1 + y2 + 4y3 \geq -3$

$5y1 - 7y2 - 6y3 = 4$

$y1, y2 \geq 0$ and $y3$ is unrestricted.

Exercise 7:

Min. $Z^* = 5y1 - y2 + 5y3$

Subject to the constraints

$y1 + y2 \geq 3$

$5y1 + y2 \geq 1$

$3y1 - y3 \geq 1$

$4y1 + y3 \geq -1$

$y1, y3 \geq 0$ and $y2$ is unrestricted.

Exercise 8 :

1. Solve the IPP as an ordinary LPP by ignoring the restriction of integer values

$$\text{Max}: Z=X_1+X_2+0S_1+0S_2$$
$$3X_1+2X_2+1S_1+0S_2=5$$

$$X_2+0S_1+1S_2=2$$

$$X_1, X_2, S_1, S_2 \geq 0, \text{ and are integers}$$

Initial Simplex Table

C_j		**1**	**1**	**0**	**0**	**B_i**	**B_i/a_{ij}**
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**(solution)**	
0	**S_1**	3	2	1	0	5	5/3
0	**S_2**	0	1	0	1	2	/
$Z=\sum_{j=1}^{n} C_j X_j$		**0**	**0**	**0**	**0**	**Z=0**	
C_j-Z_j		**1**	**1**	**0**	**0**		

Second Simplex Table

C_j		**1**	**1**	**0**	**0**	**B_i**	**B_i/a_{ij}**
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**(solution)**	
1	**X_1**	1	2/3	1/3	0	5/3	5/2
0	**S_2**	0	1	0	1	2	2
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**2/3**	**1/3**	**0**	**Z=5/3**	
C_j-Z_j		**0**	**2/3**	**-1/3**	**0**		

Third Simplex Table

C_j		1	1	0	0	B_i
C_b	X_j	X_1	X_2	S_1	S_2	(solution)
1	X_1	1	0	1/3	-2/3	1/3
1	X_2	0	1	0	1	2
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**1/3**	**1/3**	**Z=7/3**
C_j-Z_j		**0**	**0**	**-1/3**	**-1/3**	

2. Test the integrality of the optimum solution. The solution is x_1 =1/3, x_2=2 and Z=7/3. Since x_1 =1/3 is non-integer solution so that the current feasible solution is not optimal integer solution.

3. Select the source row. ⇒ The row corresponding to non-integer solution.

$x_1 + 1/3s_1 - 2/3s_2 = 1/3$

4. Find the new constraint (Gomorian constraint) from the source row.

$x_1 + 1/3s_1 - 2/3s_2 = 1/3$

$(1+0)\ x_1 + (0+1/3)\ s_1 + (-1+ 1/3)\ s_2 = 0+1/3$, and eliminate the integer part

$1/3s_1+1/3s_2 \geq 1/3$,

$-1/3s_1-1/3s_2 \leq -1/3$

$-1/3s_1-1/3s_2 + G_1= -1/3$ ⇒Gomorian constraint and G is Gomorian slack.

5. Add the new (Gomorian) constraint at the bottom of the simplex table obtained in step 1 and find the new feasible solution.

C_j		1	1	0	0	0	B_i
C_b	BV	X_1	X_2	S_1	S_2	G_1	(solution)
1	X_1	1	0	1/3	-2/3	0	1/3
1	X_2	0	1	0	1	0	2
0	G_1	0	0	-1/3	1/3	1	-1/3
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**1/3**	**1/3**	**0**	**Z=7/3**
C_j-Z_j		**0**	**0**	**-1/3**	**-1/3**	**0**	

⇒The solution is not feasible because the solution (G_1=-1/3) is –ve. Therefore, G_1 will leave the base.

- Select the new constraint as a pivot row. In order to select the entering variable, use :

$$\textbf{Max}\ \left(\frac{\Delta}{a_{ij}}, a_{ij} \leq 0\right)$$

- Therefore, $\textbf{Max}\ \left(\frac{\Delta}{a_{ij}}, a_{ij} \leq 0\right) = \textbf{Max}\left(\mathbf{1/3/\text{-}1/3,\ 1/3/\text{-}1/3}\right) = \mathbf{(-1, -1)}$

Choose arbitrarily (Let s_1 is chosen as entering variable)

C_j		1	1	0	0	0	B_i
C_b	**BV**	**X_1**	**X_2**	**S_1**	**S_2**	**G_1**	**(solution)**
1	**X_1**	1	0	0	-1	1	0
1	**X_2**	0	1	0	1	0	2
0	**S_1**	0	0	1	1	-3	1
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**0**	**0**	**1**	**Z=2**
C_j-Z_j		**0**	**0**	**0**	**0**	**-1**	

Since all $\Delta_j \geq 0$ and all $X_{bi} \geq 0$ and are integers, the current feasible solution is optimal integer solution. Therefore, the solution is x_1=0, x_2=2 and Z=2

Exercise 9:

Here Σb is greater than Σd hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

	X	Y	Z	Dummy	Availability
A	⑦ 4	③ 3	2	0	10
B	5	⑧ 6	1	0	8
C	6	① 4	④ 3	0	5
D	3	5	① 4	⑤ 0	6
Requirement.	7	12	5	5	29

For cell AX the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell AX. Next 10 – 7 = 3, this is allocated to cell AY to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to m + n – 1, then the solution is basic feasible solution. The solution, we got have 7 allocations which is = 4 + 4 – 1 = 7. Hence the solution is basic feasible solution.

Allocations are:

From	To	Units in tons	Cost in DA
A	X	7	7 × 4 = 28
A	Y	3	3 × 3 = 9
B	Y	8	8 × 6 = 48
C	Y	1	1 × 4 = 4
C	Z	4	4 × 3 = 12
D	Z	1	1 × 4 = 4
D	Dummy	5	5 × 0 = 0
	Total in DA		105

Exercise 10:

D. Centres / Plant	D_1	D_2	D_3	D_4	Av.
P_1	**19** 5	30	50	**12** 2	7
P_2	*70*	30	**40** 7	**60** 3	10
P_3	40	**10** 8	60	**20** 10	18
Requirement	5	8	7	15	35

Now we perform optimality test by applying the MODI method. First of all, we write the cost matrix for only allocated cells:

19			12	u_1
		40	60	u_2
	10		20	u_3
v_1	v_2	v_3	v_4	

Let us denote the row numbers by u_1, u_2, u_3, and column numbers by v_1,v_2,v_3, and $v_4$such that

$u_1 + v_1 = 19$, $u_1 + v_4 = 12$, $u_2 + v_3 = 40$, $u_2 + v_4 = 60$, $u_3 + v_2 = 10$, $u_3 + v_4 = 20$.

Taking u_1=0, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations. Putting the value of v_4 in the fourth equation, we have $u_2 = 48$.

Similarly, we find $u_3 = 8$, $v_2 = 2$ *and* $v_3 = -8$

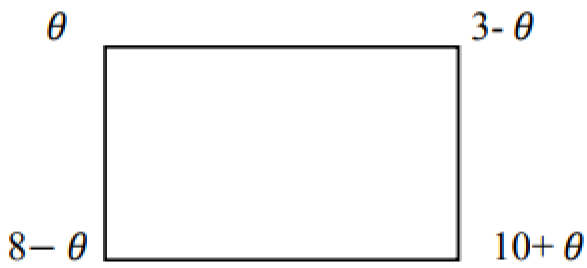
Using these values, we fill the vacant cells of the above table using $C_{ij} = u_i + v_j$ and put dots in the already filled cells so that these cells are not considered again.

	v_1= 19	v_2= 2	v_3= -8	v_4= 12
u_1= 0	•	2	-8	•
u_2= 48	67	50	•	•
u_3= 8	27	•	0	•

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij}= C_{ij} - (u_i + v_j)$

•	30-2= 28	50- (-8)= 58	•
70-67= 3	30-50= -20	•	•
40-27= 13	•	60-0= 60	•

Since Note that the cell ($P_{2,2}$)has the most negative opportunity cost (net change in cost). Therefore, the transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the closed loop formed, beginning from this cell and using allocated cells, DA20 can be saved. We form the loop beginning from this cell, i.e., the cell ($P_{2,2}$) (see Figure below). We shift θ units to this unoccupied cell through the loop and add and subtract θ from the cells at the other corners of the loop which are assigned '+' and '-' signs. So, we get :



The maximum number of the units that can be allocated to the cell (P_2,D_2) through this loop is given by the minimum of the solution of the equation 3- θ = 0 and 8- θ = 0, i.e.,

$$\theta = \min.\begin{cases} \text{the no. of units } i(P_2,D_4) = 3 \\ \text{the no. of units in } (P_{2,2}) = 8 \end{cases} = 3$$

So, with the improved allocations, the table now becomes:

19 5	30	50	12 2
70	30 3	40 7	60
40	10 5	60	20 13

Thus, the total cost of transportation for this set

=19×5+12×2+30×3+40×7+10×5+20×13

= 95+24+90+280+50+260 =799

Now, let us apply the optimality test to the improved solution. Proceeding in the same way as in the first iteration, first of all, we write the cost matrix for only allocated cells:

19			12	u_1
	30	40		u_2
	10		20	u_3
v_1	v_2	v_3	v_4	

Let us denote the row numbers by u_1, u_2, $u_3$and column numbers by v_1, v_2, v_3, and v_4 such that

$u_1 + v_1 = 19$, $u_1 + v_4 = 12$, $u_2 + v_3 = 40$,

$u_2 + v_2 = 30$, $u_3 + v_2 = 10$, $u_3 + v_4 = 20$.

Taking u_1=0, we have v_1 = 19 and v_4 = 12 from the first two equations.

Putting the value of v_4 in the fourth equation, we have u_3 = 8.

Similarly, u_2 = 8, v_2 = 2 *and* v_3 = 12

Using these values, we fill all the vacant (unoccupied) cells of the table using $C_{ij} = u_i + v_j$ for each unoccupied cell and put dots in the already filled cells so that these cells are not considered again.

	v_1= 19	v_2= 2	v_3= 12	v_4= 12
u_1= 0	•	2	-8	•
u_2= 28	47	•	•	40
u_3= 8	27	•	20	•

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

•	28	38	•
23	•	•	20
13	•	40	•

Since none of the net evaluation is negative, this solution is optimal. Thus, the total minimum transportation cost is Da 799 and the maximum saving = (1000-799) = 201.