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Chapter 7: Transportation

Methods of finding optimal solution

Once an initial basic feasible solution is determined, the next step is to check its optimality. For this, we performed an optimality test which tell us that whether the obtained feasible solution is optimal or not. This can be done by applying one of the following methods:

(a) Stepping Stone Method.

(b) Modified Distribution (MODI) Methods.

1. Stepping Stone Method.

Steps involved in Stepping Stone method for obtaining the optimal solution of a transportation problem can be summarized as follows:

1. First find an initial basic feasible solution.

2. Check optimality conditions if conditions are satisfied then evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell as follows:

a) Select an unoccupied cell to be evaluated.

b) Starting from this cell, from a closed path (or loop) through at least three occupied cells. The direction of movement is immaterial because the result will be same in both directions. Note that expect for the evaluated cell, all cells at the corners of the loop have to be occupied.

c) At each corner of the closed path, assign plus (+) and minus (-) sign alternatively; beginning with the plus sign for the unoccupied cell to be occupied.

d) Compute the net change in cost with respect to the costs associated with each cell traced in the closed path. e) Repeat steps 2(a) to 2(d) until the net change in cost has been calculated for all occupied cells.

3. If the net changes are positive or zero, an optimal solution has been arrived at. Otherwise go to step 4.

4. If some net changes are negative, select the unoccupied cell having the most negative net change. If two negative values are equal, select the one that results in moving more units into the selected unoccupied cell with the minimum cost.

5. Assign as many units as possible to this unoccupied cell. 6. Go the Step 2 and repeat the procedure until all unoccupied cells are evaluated and the value of net change, i.e., net evaluation is positive or zero.

Example 5. A company is spending DA1000 on transportation of its units from three plants to four distribution centres. The availability of unit per plant and requirement of units per distribution centre, with unit cost of transportation are given as follows:

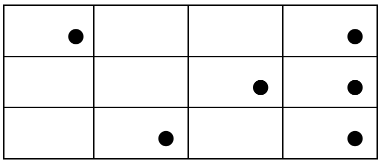
D. Centres / Plants	D_1	D_2	D_3	D_4	capacity
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirement	5	8	7	15	

What is the maximum possible saving by optimum distribution? Use the Stepping Stone method to solve the problem.

Solution: First, we determine the initial feasible solution by applying VAM is given as follows:

D. Centers / Plant	D_1	D_2	D_3	D_4	Av.	$Diff_1$	$Diff_2$	$Diff_3$	$Diff_4$
P_1	**19** (5)	30	50	**12** (2)	7	7	18	38	38
P_2	*70*	30	**40** (7)	**60** (3)	10	10	10	20	20
P_3	40	**10** (8)	6	**20** (10)	18	10	10	40^*	-
Requirement	5^*	8	7	15					
$Diff_1$	21^*	20	10	8					
$Diff_2$	-	20^*	10	8					
$Diff_3$	-	-	10	8					
$Diff_4$	-	-	10	48^*					

In the above table, the encircled values are the allocations. The total transportation cost associated with this initial basic feasible solution is = 19 × 5 + 12 × 2 + 40 × 7 + 60 × 3 + 10 × 8 + 20 × 10 = 95 + 24 + 280 + 180 + 80 + 200 = 859 Clearly, here we have 3+4-1= 6 occupied cells. These are independent as it is not possible to form any closed loop through these allocations (see Fig below.)



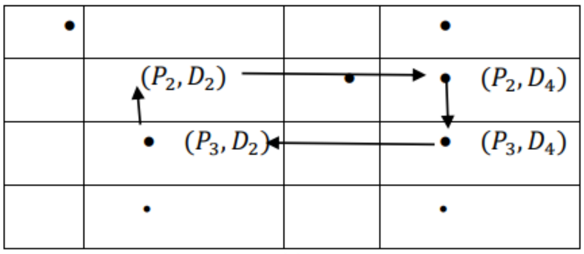
Hence, all the conditions of applying optimality test is satisfied so we apply Stepping Stone method to obtain optimal solution.

Optimality Test using the Stepping Stone Method

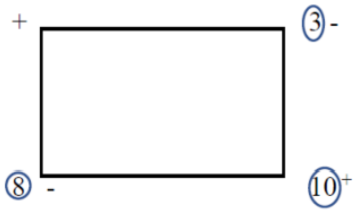
We evaluate the effect of allocating one unit to each of the unoccupied cells making closed paths. Note that the unoccupied cells are (P_1,D_2), (P_1,D_3), (P_2,D_1), (P_2,D_2), (P_3,D_1) and (P_3,D_3). We have to make closed paths so that each path contains at least three occupied cells. We also have to evaluate the net change in cost for each and every unoccupied cell. Then we have to select the one unoccupied cell, which has most negative opportunity cost and allocate as many units as possible to reduce the total transportation cost. The computations are shown as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Da)
(P_1,D_2)	$(P1,2) \rightarrow (P1,4) \rightarrow (P3,D4) \rightarrow (P3,D2)$	30-12+20-10 = 28
(P_1,D_3)	$(P1,3) \rightarrow (P1,4) \rightarrow (P2,D4) \rightarrow (P2,D3)$	50-12+60-40 = 58
(P_2,D_1)	$(P2, D1) \rightarrow (P1,1) \rightarrow (P1,D4) \rightarrow (P2,D4)$	70-19+12-60 = 3
(P_2,D_2)	$(P2,2) \rightarrow (P3,2) \rightarrow (P3,D4) \rightarrow (P2,D4)$	30-10+20-60 = -20
(P_3,D_1)	$(P3, D1) \rightarrow (P1,1) \rightarrow (P1,D4) \rightarrow (P3,D4)$	40-19+12-20 = 13
(P_3,D_3)	$(P3,3) \rightarrow (P2,3) \rightarrow (P2,D4) \rightarrow (P3,D4)$	60-40+60-20 = 60

The cell $(P2,2)$ has the most negative opportunity cost (net change in cost). Therefore, transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the loop shown in the fourth row of the table, then Rs20 can be saved (see Fig.4.3a). hence, we shall shift as many units as possible to the cell $(P2,D2)$through this loop. The maximum number of units that can be allocated to $(P2,2)$ through this loop is 3.



(a)



(b)

This is because the shifting can be done only from the corners of the loop and more than 3 units cannot be shifted to $(P2,2)$as explained below: The allocations at the corners of the loops are 3,10 and 8. If we try to shift more than 3 units, say, 4 units from the corner $(P2,4)$, then 4 units will have to be subtracted from the corner $(P3,D2)$ so that the total of the column D4 remains unchanged. But this will give 3-4 = -1 allocations to the cell $(P2,4)$, which is impossible as negative allocations cannot be made. We obtain the maximum number of units that can be allocated to the cell $(P2,2)$ through the loop as follows:

1. First, we assign (+) sign to the unoccupied cell $(P2,2)$ to be evaluated and then (-) and (+) signs alternatively to other corners of the closed loop (moving in one direction) as shown in figure above.
2. Then we take the minimum of the values at the corners that have been assigned the negative sign.

In this case, the maximum number of units that can be allocated to the cell $(P2,2)$ through the mentioned loop is the minimum of 3 and 8. It is 3. So, we write it as:

$$\min. \begin{cases} the\ no.\ of\ units\ in (P2,D4) = 3 \\ the\ no.\ of\ units\ in\ (P2,2) = 8 \end{cases} = 3$$

The new table with these changes becomes:

	D_1	D_2	D_3	D_4
P_1	19 (5)	30	50	(2) 12
P_2	70	30 (3)	40 (7)	60
P_3	40	10 (5)	60	(13) 20

In the above table, note that we have also allocated 3 units from (P3, D2) to (P3, D4) so that column D4 remains unchanged. This leaves 5 units in the cell (P3, D2) and there are13 units in (P3, D4). The total transportation cost associated with this solution is

Total cost =19 × 5 + 12 × 2 + 30 × 3 + 40 × 7 + 10 × 5 + 20 × 13

= 95 + 24 + 90 + 280 + 50 + 260 = 799

Now, we repeat the optimality test to see if further allocation can be made to reduce the total transportation cost. The computation for the unoccupied cells is as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Da)
(P_1,D_2)	$(P1,2) \rightarrow (P1,4) \rightarrow (P3,D4) \rightarrow (P3,D2)$	30-12+20-10 = 28
(P_1,D_3)	$(P1,D3) \rightarrow (P1,D4) \rightarrow (P3,D4) \rightarrow (P3,D2) \rightarrow (P2,D2) \rightarrow (P2,D3)$	50-12+20-10+30-40 = 38
(P_2,D_1)	$(P2, D1) \rightarrow (P1,D1) \rightarrow (P1,D4) \rightarrow (P3,D4) \rightarrow (P3,D2) \rightarrow (P2, D2)$	70-19+12-20+10-30 = 23
(P_2,D_2)	$(P2,4) \rightarrow (P3,4) \rightarrow (P3,D2) \rightarrow (P2,D2)$	60-20+10-30 = 20
(P_3,D_1)	$(P3, D1) \rightarrow (P1,1) \rightarrow (P1,D4) \rightarrow (P3,D4)$	40-19+12-20 = 13
(P_3,D_3)	$(P3,3) \rightarrow (P2,3) \rightarrow (P2,D2) \rightarrow (P3,D2)$	60-40+30-10 = 40

Since, all opportunity costs in the unoccupied cells are non-negative, the current solution is an optimal solution with total transportations cost 799. Hence the maximum saving by optimum distribution is (1000-799) = 201.

2. Modified Distribution (MODI) Methods.

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are represented by u_i, i=1,2,…..m where as the dual variables corresponding to the column constraints are represented by v_j, j=1,2,…..n. The values of the dual variables are calculated from the equation given below

$u_i + v_j = c_{ij}$ if $x_{ij} > 0$

Step 3: Any basic feasible solution has m + n -1 $x_{ij} > 0$. Thus, there will be m + n -1 equation to determine m + n dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If $x_{ij}=0$, the dual variables calculated in Step 3 are compared with the c_{ij} values of this allocation as $c_{ij} – u_i – v_j$. If al $c_{ij} – u_i – v_j \geq 0$, then by the **theorem of complementary slackness** it can be shown that the corresponding solution of the transportation problem is optimum. If one or more $c_{ij} – u_i – v_j < 0$, we select the cell with the least value of $c_{ij} – u_i – v_j$ and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Example 6: For example, consider the transportation problem given below:

Supply

	1	9	13	36	51	50
	24	12	16	20	1	100
	14	33	1	23	26	150
Demand	100	70	50	40	40	300

Solution:

Step 1: First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

$x_{11}=50$, $x_{22}=60$, $x_{25}=40$, $x_{31}=50$, $x_{32}=10$, $x_{33}=50$ and $x_{34}=40$

Step 2: The dual variables u_1, u_2, u_3 and v_1, v_2, v_3, v_4, v_5 can be calculated from the corresponding c_{ij} values, that is

$u_1+v_1=1$ $u_2+v_2=12$ $u_2+v_5=1$ $u_3+v_1=14$

$u_3+v_2=33$ $u_3+v_3=1$ $u_3+v_4=23$

Step 3: Choose one of the dual variables arbitrarily is zero that is $u_3=0$ as it occurs most often in the above equations. The values of the variables calculated are:

$u_1= -13$, $u_2= -21$, $u_3=0$

$v_1=14$, $v_2=33$, $v_3=1$, $v_4=23$, $v_5=22$

Step 4: Now we calculate $c_{ij} – u_i – v_j$ values for all the cells where $x_{ij}=0$ (.e. unallocated cell by the basic feasible solution)

That is

Cell(1,2)= c12-u1-v2 = 9+13-33 = -11

Cell(1,3)= c13-u1-v3 = 13+13-1 = 25

Cell(1,4)= c14-u1-v4 = 36+13-23 = 26

Cell(1,5)= c15-u1-v5 = 51+13-22 = 42

Cell(2,1)= c21-u2-v1 = 24+21-14 = 31

Cell(2,3)= c23-u2-v3 = 16+21-1 = 36

Cell(2,4)= c24-u2-v4 = 20+21-23 = 18

Cell(3,5)= c35-u3-v5 = 26-0-22 = 4

Note that in the above calculation all the $c_{ij} – u_i – v_j \geq 0$ except for cell (1, 2) where $c_{12} – u_1 – v_2 =$ 9+13- 33 = -11.

Thus in the next iteration x_{12} will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is

-33 -1 + 9 +14 = -11

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$x_{11}=40$, $x_{12}=10$, $x_{22}=60$, $x_{25}=40$, $x_{31}=60$, $x_{33}=50$, $x_{34}=40$