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Chapter 6: Integer Programming: Cutting-Plane Method

Integer LP problems are those in which some or all of the variables are restricted to integer (or discrete) values. An integer LP problem has important applications. Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, batch size, capacity expansion, fixed charge, etc., are few problems that demonstrate the areas of application of integer programming.

Types of integer programming problems

Linear integer programming problems can be classified into three categories:

A. **Pure (all) integer programming problems** in which all decision variables are restricted to integer values. For example:

$$\text{Max}: Z=3X_1+2X_2$$
$$\text{S.T} \quad X_1+X_2 \leq 12$$
$$X_1, X_2 \geq 0, X_1, X_2 \text{ integer}$$

B. **Mixed integer programming problems** in which some, but not all, of the decision variables are restricted to integer values.

For example,

$$\text{Max}: Z=3X_1+2X_2$$
$$\text{S.T} \quad X_1+X_2 \leq 12$$
$$X_1, X_2 \geq 0, X_1 \text{ integer}$$

The LP above is a mixed integer programming problem (X_2 is not required to be an integer)

C. Zero-one integer programming problems in which all decision variables are restricted to integer values of either 0 or 1.

$$\text{Max}: Z=X_1-X_2$$
$$\text{S.T} \quad X_1+2X_2 \leq 2$$
$$2X_1-X_2 \leq 1$$
$$X_1, X_2=0 \text{ or } 1$$

1. Enumeration and Cutting-Plane Algorithm

The cutting-plane algorithm starts at the continuous optimum LP solution. Special constraints (called cuts) are added to the solution space in a manner that renders an integer optimum extreme point. The cutting-plane method to solve integer LP problems was developed by R.E. Gomory in 1956. This method is based on creating a sequence of linear inequalities called cuts. Such a cut reduces a part of the feasible region of the given LP problem, leaving out a feasible region of the integer LP problem. The hyperplane boundary of a cut is called the cutting plane.

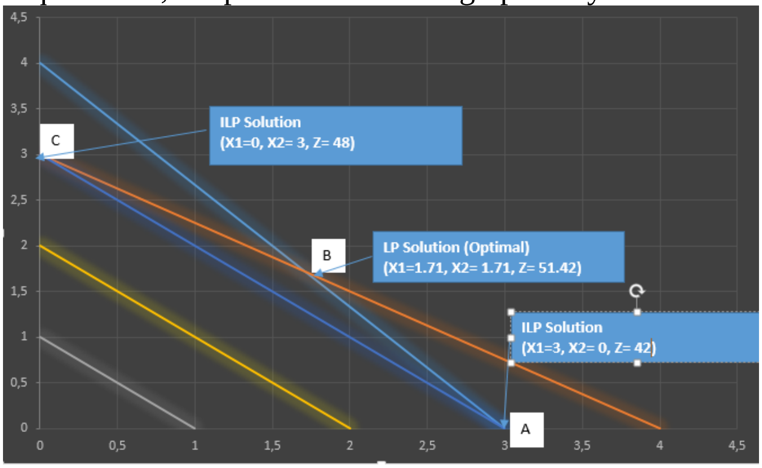
Example1: Consider the following linear integer programming (LIP) problem

$$\text{Max}: Z=14X_1+16X_2$$
$$\text{S.T} \quad 4X_1+3X_2 \leq 12$$
$$6X_1+8X_2 \leq 24$$
$$X_1, X_2 \geq 0, \text{ and are integers}$$

Solution

In Example 1, we first demonstrate graphically how cuts are used to produce an integer solution and then implement the idea algebraically**.**

Relaxing the integer requirement, the problem is solved graphically as shown in Figure below.



The optimal solution to this LP problem is: x_1 = 1.71, x_2 = 1.71 and Max Z = 51.42. This solution does not satisfy the integer requirement of variables x_1 and x_2.

Rounding off this solution to x_1 = 2, x_2 = 2 does not satisfy both the constraints and therefore, the solution is infeasible. The dots in Figure above, also referred to as lattice points, represent all of the integer solutions that lie within the feasible solution space of the LP problem. However, it is difficult to evaluate every such point in order to determine the value of the objective function.

In Figure above, it may be noted that the optimal lattice point C, lies at the corner of the solution space OABC, obtained by cutting away the small portion above the dotted line. This suggests a solution procedure that successively reduces the feasible solution space until an integer-valued corner is found. The optimal integer solution is: x_1 = 0, x_2 = 3 and Max Z = 48. The lattice point, C is not even adjacent to the most desirable LP problem solution corner, B.

Remark: Reducing the feasible region by adding extra constraints (cut) can never give an improved objective function value. If Z_{1P} represents the maximum value of objective function in an ILP problem and Z_{LP} the maximum value of objective function in an LP problem, then $Z_{1P} \leq Z_{LP}$.

2. Gomory's all integer Cutting-Plane method

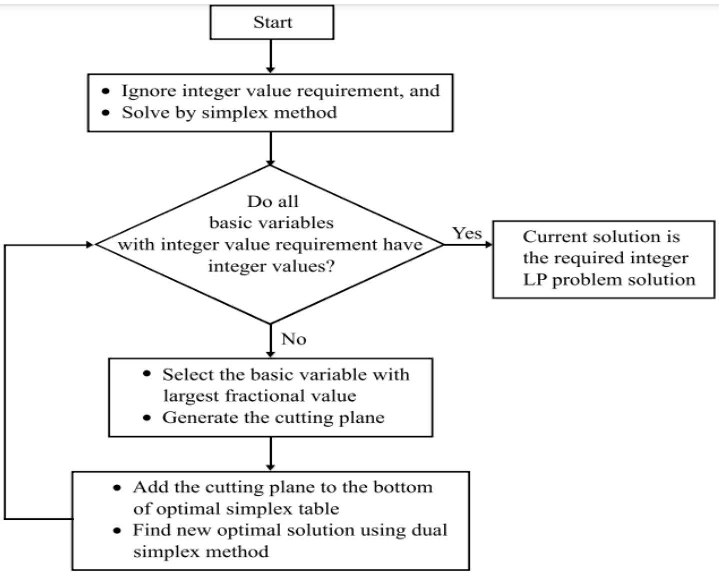
Gomory's algorithm has the following properties:

(a) Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.

(b) Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.

Steps of Gomory's all integer Cutting-Plane method:

The procedure for solving an ILP problem is summarized in a flow chart shown in Figure below



Example2: Solve the following Integer LP problem using Gomory's cutting plane method.

Max : $Z=X_1+X_2$
S.T $3X_1+2X_2 \leq 5$

$X_2 \leq 2$

$X_1, X_2 \geq 0$, and are integers

Solution:

Step 1: Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.

C_j		**1**	**1**	**0**	**0**	**B_i**
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**(solution)**
1	**X_1**	1	0	1/3	-2/3	1/3
1	**X_2**	0	1	0	1	2
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**1/3**	**1/3**	**Z=7/3**
C_j-Z_j		**0**	**0**	**-1/3**	**-1/3**	

In Table above, since all cj – zj ≤ 0, the optimal solution of LP problem is: $x_1 = 1/3$, $x_2 = 2$ and Max Z = 7/2.

Step 2: In the current optimal solution, shown in Table above x_1 did not assume integer value. Thus, solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

Step 3: Since x1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (x_1-row) as source row in Table above to generate Gomory cut as follows:

$1/3 = x_1 + 0.x_2 + 1/3\ s_1 – 2/3\ s_2$ (x_1-source row)

The factoring of numbers (integer plus fractional) in the x_1-source row gives

$$\left(0+\frac{1}{3}\right) = (1 + 0)\, x_1 + \left(0+\frac{1}{3}\right) s_1 + \left(-1+\frac{1}{3}\right) s_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

Rearranging all of the integer coefficients on the left-hand side, we get

$1/3 + (s_2 – x_1) = 1/3\ s_1 + 1/3\ s_2$

Since value of variables x_1 and s_2 is assumed to be non-negative integer, left-hand side must satisfy $1/3 \leq 1/3\ s_1 + 1/3\ s_2$

$1/3 + s_{g1} = 1/3\ s_1 + 1/3\ s_2$ or $s_{g1} – 1/3\ s_1 – 1/3\ s_2 = – 1/3$ (Cut I)

where s_{g1} is the new non-negative (integer) slack variable. Adding this equation (also called Gomory cut) at the bottom of Table above, the new values so obtained is shown in Table below.

C_j		**1**	**1**	**0**	**0**	**0**	**B_i** (solution)
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**S_{g1}**	
1	**X_1**	1	0	1/3	-2/3	0	1/3
1	**X_2**	0	1	0	1	0	2
0	**S_{g1}**	0	0	-1/3	-1/3	1 ➡	-1/3
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**1/3**	**1/3**	**0**	**Z=7/3**
C_j-Z_j		**0**	**0**	**-1/3**	**-1/3**	**0**	
Ratio : Min (C_j-Z_j)/Y_{3j} (<0)		**-**	**-**	**1** ⬆	**1**	**-**	

Step 4: Since the solution shown in Table above is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table above The new solution is obtained by applying the following row operations:

$R_3(\text{new}) \rightarrow R_3(\text{old}) \times –3$; $R_1(\text{new}) \rightarrow R_1(\text{old}) – (1/3)\ R_3(\text{new})$

The new solution is shown in Table below:

C_j		**1**	**1**	**0**	**0**	**0**	**B_i** (solution)
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**S_{g1}**	
1	**X_1**	1	0	0	-1	1	0
1	**X_2**	0	1	0	1	0	2
0	**S_{g1}**	0	0	1	1	-3	1
$Z=\sum_{j=1}^{n} C_j X_j$		**1**	**1**	**0**	**0**	**1**	**Z=2**
C_j-Z_j		**0**	**0**	**0**	**0**	**-1**	

Since all $c_j – z_j \leq 0$ and value of basic variables shown in x_B-column of Table above is integer, the solution: $x_1 = 0$, $x_2 = 2$, $s_{g1} = 1$ and Max $Z = 2$, is an optimal basic feasible solution of the given ILP problem.