Prepared by: Dr. Lemya Mekarssi. **Level: 2nd Year Bachelor of Business Sciences**

Sensitivity analyses

The importance of sensitive analysis lies in that it gives a full study of the variables included in the mathematical model so that we remain with the largest return or the lowest cost and what is the extent of changes in these variables, for example if changes occur in the conditions of the project such as an increase in the available resources such as an increase in the available time or in the number of workers or the production of a new product or otherwise, which requires the re-solution of the model to the problem after adding the new variables.

The question of resolving the model is cumbersome and may take a long time, but applying the sensitivity analysis method (also called post-optimization analysis) avoids resolving the whole solution.

Sensitivity analysis is the study of the effect of changes in problem components on a linear programming model.

We distinguish the following cases of the most important changes that can occur on the initial linear programming model:

- Case of changing the coefficients of the objective function (Cj);
- Case of changing the right side of the constraints (bi);
- Case of changing the technical coefficients (aij);
- Adding a new variable or variables;
- Adding a new constraint or constraints.

These variables give rise to one of the following conditions:

- The optimal solution does not change (the optimal solution is not affected by the new changes).
- Basic variables do not change as their values change as a result of new additional changes.
- The whole basic solution has changed.

Note: Sensitivity analysis does not require solving the problem from the beginning, but only by studying the optimal table (the optimal solution).

In this lecture, the focus will be on studying changes in the coefficients of the objective function and changes in the right side of constraints.

Example: Here is the following linear program:

$$\text{Max}: Z=2X_1+X_2$$
$$3X_1+X_2\leq 15$$
$$X_1+2X_2\leq 12$$
$$X_1\geq 0,\ X_2\geq 0$$

If you know that, the optimal solution for this model reached in the optimal simplex table is:

C_b →		**2**	**1**	**0**	**0**	**B_i**	**B_i/a_{ij}**
↓		**X_1**	**X_2**	**S_1**	**S_2**		
2	**X_1**	1	0	6/15	-1/5	18/5	
1	**X_2**	0	1	-1/5	3/5	21/5	
$Z=\sum_{j=1}^{n} C_j X_j$		**2**	**1**	**9/15**	**1/5**	**Z=(2*18/5)+(1*21/5)=32**	
C_j-Z_j		**0**	**0**	**-9/15**	**-1/5**		

1. **Sensitivity Analysis on Objective Function Coefficients:**

One of the most important things that decision-makers are concerned with is the sensitivity of linear programming to changes that may affect prices. A change in the coefficients of the objective function will affect the optimality of the solution reached. In this context, when conducting a sensitivity analysis, it is necessary to determine, whether the change in the coefficients of the basic objective function only, or the non-basic variables, or both.

Suppose that the coefficient Cj changes within the range from 2 to 2+Δ, and by substituting the amount 2 with the amount 2+Δ in the last simplex table, we get the following table:

C_b →		$2+\Delta$	1	0	0	B_i	B_i/a_{ij}
		X_1	X_2	S_1	S_2		
$2+\Delta$	X_1	1	0	6/15	-1/5	18/5	
1	X_2	0	1	-1/5	3/5	21/5	
$Z=\sum_{j=1}^{n} C_j X_j$		$2+\Delta$	1	$\frac{9+6\Delta}{15}$	$\frac{1-\Delta}{5}$	$Z=\frac{57+18\Delta}{5}$	
C_j-Z_j		0	0	$\frac{-(9+6\Delta)}{15}$	$\frac{-(1-\Delta)}{5}$		

The solution given in the table remains an ideal solution if:

$$\begin{cases} \frac{-(9+6\Delta)}{15} \leq 0 \\ \frac{-(1-\Delta)}{5} \leq 0 \end{cases} \Leftrightarrow \begin{cases} \Delta \geq -3/2 \\ \Delta \leq 1 \end{cases} \Leftrightarrow -3/2 \geq \Delta \geq 1$$

This means that the optimal solution will take the value (18/5, 21/5) = (X_1, X_2) whenever:

$$0.5 \leq C1 \leq 3$$

If the changes that may affect the parameter C1 are within the range (range) **[3, 0.5]** that has been defined using sensitivity, there will be no effect on the optimal solution. However, if the changes are outside the specified range, a new solution is needed and the problem must be reprogrammed.

2. Sensitivity Analysis on Coefficients bi:

Assuming that there are changes in the value of the available resources bi in the first constraint of the previous example, and let us assume that this coefficient changes within the range from 15 to 15+Δ and by substituting the amount 15 with the amount 15+Δ, in the first constraint only in the initial solution table: constraint only in the initial solution table**:**

C_j		2	1	0	0	B_i
C_b	X_j	X_1	X_2	S_1	S_2	
0	S_1	3	1	1	0	15+1 **Δ**
0	S_2	1	2	0	1	12+0 **Δ**
$Z=\sum_{j=1}^{n} C_j X_j$		0	0	0	0	**Z=0**
C_j-Z_j		2	1	0	0	

In the last simplex table (the optimal solution), the column opposite: Si gives us the coefficients of **Δ** in the column of quantities (available), we get the following table:

C_b →		2	1	0	0	
		X_1	X_2	S_1	S_2	B_i
2	X_1	1	0	6/15	-1/5	$\frac{54+6\Delta}{15}$
1	X_2	0	1	-1/5	3/5	$\frac{21-\Delta}{5}$
$Z=\sum_{j=1}^{n} C_j X_j$		2	1	**9/15**	**1/5**	**Z=32+9/15Δ**
C_j-Z_j		0	0	**-9/15**	**-1/5**	

We obtained the solution values bi by entering the value Δ multiplied by the coefficients of Si in the last column of the optimal solution table.
The value (6/15) means that reducing S_1 by one unit will result in an increase in X_1 by (6/15) units, and vice versa. By the same logic, it is possible to analyse for X_2.
The solution given in the table remains an optimal solution if:

$$\begin{cases} \frac{54+6\Delta}{15} \geq 0 \\ \frac{21-\Delta}{5} \geq 0 \end{cases} \Leftrightarrow \begin{cases} \Delta \geq -9 \\ \Delta \leq 21 \end{cases} \Leftrightarrow -9 \geq \Delta \geq 21$$

This means that the optimal solution will be stable whenever:

$$(15-9) \leq b_1 \leq (15+21) \Leftrightarrow 6 \leq b_1 \leq 36$$

3. Sensitivity Analysis on Technical Coefficients a_{ij}:

It may happen that the technological coefficients change because their values are originally largely determined by the type of technology used (for example, obsolescence, repair or replacement of machines with better performance). Here, the decision-maker is interested in the extent to which this change affects the optimal solution. In general, changes in the coefficients of decision variables will have a direct impact on the elements of the solution matrix, which complicates the calculations and can affect the left-hand side of the constraints.
It is important when making a change to the technical coefficient a_{ij} to know whether these coefficients correspond to a basic variable or a non-basic variable in the optimal solution. Here we distinguish between two cases:

- If this variable X_j is a non-basic variable, this means that the change that occurs in the technical coefficient will only affect its coefficient: C_j-Z_j.

- If this variable X_j is a basic variable, this means that the change that occurs in the technical coefficient will only affect its coefficient: C_j-Z_j, and the values of the basic variables.