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Chapter 5: Linear Programming: Duality

1. Definition of the Dual Program:

Dual programs (Duality) emerged due to the difficulty of solving the original programs, and the dual program or model can be defined as the formula corresponding to the formula of the original program. That is, for each of the linear programming models, there is a corresponding and derived model, so if the first model is related to maximizing the objective function, the corresponding model will be minimizing the objective function and is usually formulated from the same data contained in the first model and vice versa.

Remark: The original problem is usually referred to as primal.

2. Features of the corresponding model:

- The corresponding model helps to reach a solution faster in some cases by reducing the solution steps, meaning that the method of solving the corresponding problem requires less complex mathematical steps than the steps required to solve the original problem sometimes.
- To get rid of the negative sign on the right side (if any)
- For the purpose of identifying the dimensions of the other problem (the dual, alternative problem) If the original model is in the form of Max, meaning the problem in the profitable formula, we can identify the dual model and it is in the form of Min and represents it for the cost side (in the same problem), and for the same problem expressed first in the original formula.
- The number of variables in the dual model represents the number of constraints in the original model, as the number of constraints in the dual model represents the number of variables in the original model.
- The value of the objective function for solving the dual model is equal to the value of the objective function for solving the original model.
- The values of the Dj indicators corresponding to the difference variables in the final solution of the original program are equal to the values of the solution variables in the dual program.

3. Steps for Converting Any Primal into Its Dual

To convert the original program to a dual program, follow the steps shown in the following table:

Step 1: First, convert the objective function to maximization form.

Step 2: If a constraint has inequality sign $\geq$, then convert the inequality sign $\geq$ into $\leq$ on multiply both sides by -1.

Step 3: If a constraint has an equality sign (=) then, it is replaced by two constraints, one constraint having inequality sign $\geq$ and other one having $\leq$ and hence corresponding to one equality, we obtain two inequalities.

Step 4: Every unrestricted variable is replaced by the difference of two non-negative variables. In this way, we get the standard form of the given primal. Now the dual of the given problem is obtained by:

Step 5: Transposing the rows and columns of constraints coefficients. Also transposing the coefficients ($c_1, c_2, \ldots, c_n$) of the objective function and the right-side constants ($b_1, b_2, \ldots , b_m$). Changing the inequalities from $\leq$ to $\geq$ sign and minimizing the objective function instead of maximizing it.

Note: If we have an equality constraint in primal then its corresponding dual variables must be unrestricted in sign.

Result 1. The dual of the dual is primal.

Result 2. If either the primal or the dual problem has an unbounded solution, the other one has no feasible solution.

Result 3. If either the primal or the dual problem has a finite optimal solution, the other problem also has finite optimal solution. The optimal value of the objective functions of the two problems are equal.

The steps of obtaining dual of an LPP are explained with the help of an example.

Example 1. Find the dual of the following LPP.

$$\text{Max : } Z=3X_1+6X_2+8X_3$$
$$2X_1+3X_2+X_3\leq 6$$
$$5X_1+3X_2+2X_3=2$$
$$X_1\geq 0,\ X_2\geq 0,\ X_3\geq 0$$

Solution: The dual program for the linear program is written as follows:

$$\text{Min : } Z=6y_1+2y_2$$
$$2y_1+5y_2\geq 3$$
$$3y_1+3y_2\geq 6$$
$$1y_1+2y_2\geq 8$$
$$y_1\geq 0,$$
$$y_2 \text{ unrestricted sign}$$

Example 2. Find the dual of the following LPP

$$\text{Min : } Z=2X_1+4X_2+7X_3$$
$$2X_1+7X_2+X_3\leq 12$$
$$X_1+5X_2+2X_3=15$$
$$3X_1+2X_2+2X_3\leq 20$$
$$X_1\geq 0,\ X_2\geq 0,$$
$$X_3 \text{ unrestricted sign}$$

Solution: The dual program for the linear program is written as follows:

$$\text{Max : } Z=12y_1+15y_2+20y_3$$
$$2y_1+y_2+3y_3\leq 2$$
$$7y_1+5y_2+2y_3\leq 4$$
$$1y_1+2y_2+2y_3=7$$
$$y_1\leq 0$$
$$y_2 \text{ unrestricted sign}$$
$$Y_3\leq 0$$