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Chapter 4: Linear Programming: The Simplex Method

<u>Special Cases in Simplex</u>

There are some special cases that we may face when solving various linear programs in the simplex method. Among these cases, the following will be addressed:

1. Lack of an optimal solution:

There are some linear programming problems where the optimal solution is not possible, often due to an error in the composition and presentation of the constraints.

Example: Solving the following linear program using the simplex method

$$\text{Max}: Z=3X_1-2X_2$$

S / T
$$\begin{cases} X_1+X_2\leq 1 \\ 2X_1+2X_2\geq 4 \\ X_1\geq 0,\ X_2\geq 0 \end{cases}$$

Solution:

1. **Converting the program to the standard format:**

$$\text{Max}: Z=3X_1-2X_2+0S_1+0S_2-Ma_1$$
$$X_1+X_2+1S_1+0S_2+0a_1=1$$
$$2X_1+2X_2+0S_1-1S_2+1a_1=4$$
$$X_1\geq 0,\ X_2\geq 0,\ S_1\geq 0,\ S_2\geq 0,\ a_1\geq 0$$

Initial Simplex Table

C_j		3	-2	0	0	-M	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2	a_1		
0	S_1	1	1	1	0	0	1	1/1= 1
-M	a_1	2	2	0	-1	1	4	4/2= 2
$Z=\sum_{j=1}^{n} C_j X_j$		-2M	-2M	0	0	-M	Z=-4M	
C_j-Z_j		3+2M	-2+2M	0	0	0		

Incoming column

Pivot element

Second Simplex Table

C_j		3	-2	0	0	-M	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2	a_1	قيم الحل	
3	X_1	1	1	1	0	0	1	/
-M	a_1	0	0	-2	-1	1	2	/
$Z=\sum_{j=1}^{n} C_j X_j$		3	3	3+2M	+M	-M	Z=3-2M	
C_j-Z_j		0	-5	-3-2M	-M	0		

We note from the values of the last line that all values are either negative or equal to zero, and therefore according to the rule, the optimal solution has been reached, but this solution includes the artificial variable a_1, which indicates that this linear program is insoluble.

2. The case of unlimited solution:

It is the case where all the elements of the incoming column are **less than or equal to zero**, which means that it is difficult to determine the outgoing row because it is impossible to choose the variable that comes out of the base.

Example

- The second simplex table:

C_j		6	9	0	0	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2		
9	X_2	-1	1	2	0	30	/
0	S_2	0	0	-1	1	10	/
$Z=\sum_{j=1}^{n} C_j X_j$		-9	9	18	0	Z=270	
C_j-Z_j		15	0	-18	0		

Incoming column

We note from the Simplex table that all the elements of the incoming column, which corresponds to the largest positive value, are less than or equal to zero, which makes it impossible to determine the variable that comes out of the base.

3. The case of multiple solutions

It means that there is more than one solution to the problem of linear programming. When it comes to the simplex table, we are dealing with so-called alternative solutions, which means obtaining the same value of the objective function and alternative values of variables.

Example: Solving the following linear program using the simplex method

$$\text{Max}: Z=2X_1+4X_2$$

S/T

$$\begin{cases} X_1+2X_2\leq 5 \\ X_1+X_2\leq 4 \\ X_1\geq 0,\ X_2\geq 0 \end{cases}$$

Solution:

1. Converting the program to the standard form:

$$\text{Max}: Z=2X_1+4X_2+0S_1+0S_2$$
$$X_1+2X_2+1S_1+0S_2=5$$
$$X_1+X_2+0S_1-1S_2=4$$
$$X_1\geq 0,\ X_2\geq 0,\ S_1\geq 0,\ S_2\geq 0$$

Initial Simplex table

C_j		2	4	0	0	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2		
0	S_1	1	2	1	0	5	5/2
0	S_2	1	1	0	1	4	4/1
$Z=\sum_{j=1}^{n} C_j X_j$		0	0	0	0	Z=0	
C_j-Z_j		2	4	0	0		

Incoming column

We note from the values of the last line that this solution can be improved, the incoming variable is X_2, the outgoing variable is S_1.

Second Simplex Table

C_j		2	4	0	0	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2		
4	X_2	1/2	1	1/2	0	5/2	/
0	S_2	1/2	0	1/2	1	3/2	/
$Z=\sum_{j=1}^{n} C_j X_j$		2	4	2	0	Z=10	
C_j-Z_j		0	0	-2	0		

From the values of the last line of the simplex table, we note that the optimal solution has been reached as all values are either positive or equal to zero, but this solution has an alternative solution where X_1 can replace S_2.

Third Simplex table

C_j		2	4	0	0	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2		
4	X_2	0	1	0	-1	1	/
2	X_1	1	0	1	2	3	/
$Z=\sum_{j=1}^{n} C_j X_j$		2	4	2	0	Z=10	
C_j-Z_j		0	0	-2	0		

This table represents an alternative to the previous solution.

4. Status of breakeven in the values of the last line

The values of the last line in the Simplex table may be equal (equal to positive values in the case of maximization, or equal to negative values in the case of minimization), which means that it is possible to choose a variable from among the variables nominated for entry arbitrarily, and choosing any of these variables will lead to the same optimal solution. One of the following rules can be relied upon:

- **Rule 1:** If the tie in the last line is between the decision variable and the dummy variable, it is preferable in this case to choose the decision variable.

-**Rule 2:** If the parity in the last line is between two decision variables, it is preferable in this case to choose the decision variable that has a greater impact on the objective function.

- **Rule 3:** If the parity in the last line is between two dummy variables, in this case, any variable is randomly selected.

5. Condition of disintegration

There are several cases of disintegration, including:

a. The state of the equivalence of the values of (b_i/a_{ij}) when determining the variable that will come out of the solution

To illustrate this case, this example will be addressed:

Example: Solving the following linear program using the simplex method

$$\text{Max}: Z=80X_1+70X_2$$

$$\begin{cases} 2X_1+X_2 \leq 120 \\ X_1 \leq 70 \\ X_1+X_2 \leq 60 \\ X_1 \geq 0,\ X_2 \geq 0 \end{cases}$$

1.Converting the program to the standard form:

Max: $Z=80x_1+70X_2+0S_1+0S_2+0S_3$

$2X_1+ X_2+1S_1+ 0S_2+ 0S_3=120$

$X_2+0S_1 + 1S_2+ 0S_3=70$

$X_1+X_2+0S_1+ 0S_2+ 1S_3=60$

$X_1\geq0,\ X_2\geq0,\ S_1\geq0,\ S_2\geq0,\ S_3\geq0$

Initial Simplex table

C_j		**80**	**70**	**0**	**0**	**0**	**B_i**	**B_i/a_{ij}**
C_b	**X_j**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**		
0	**S_1**	2	1	1	0	0	120	60
0	**S_2**	1	0	0	1	0	70	70
0	**S_3**	1	1	0	0	1	60	60
$Z=\sum_{j=1}^{n} C_j X_j$		**0**	**0**	**0**	**0**	**0**	**Z=0**	
C_j-Z_j		**80**	**70**	**0**	**0**	**0**		

Incoming column

We note from the values of the last line in the simplex table that the optimal solution has not been reached for the existence of positive values, and the incoming variable has been determined, which corresponds to the largest positive value, and therefore the variable entering the base is X_1, and to determine the outgoing variable from the base, it is necessary to calculate the value **B_i/a_{ij}** and choose the smallest quotient, but we note a tie between the variables of the difference S_1 and S_2, both of which can be taken out of the base.

b. Recurrence of registration

The solution is considered disjointed if one of the restrictions is redundant or duplicated and therefore is not originally necessary for the solution. For example:

Constraint: $X_1 \geq 20$ No need when the constraint Appears $X_1 \geq 30$, because this last constraint removed the first constraint and became unnecessary.