Prepared by: Dr. Lemya Mekarssi. **Level: 2nd Year Bachelor of Business Sciences**

Chapter 4: Linear Programming: The Simplex Method

The Simplex method is one of the most important methods used in solving linear programs in the mathematics of the institution and operations research. It was invented and developed by researcher George Dantzig in 1947, and it is known as a highly efficient mathematical method in extracting optimal solutions to linear programming problems in general. This method is used to solve mathematical models of linear programming algebraically, regardless of the number of variables, and it is the most used to solve mathematical models .

1. Steps to apply the Simplex method:

- ✓ Converting the linear program to the model (standard) format.
- ✓ Formation of the first basic solution table (initial) allowed (it is a corner point in the possible area and often it is the zero point).
- ✓ Testing the identity of the solution (the possibility of improving the existing solution).
- ✓ Optimizing the solution until an optimal solution or a special case is reached.

Note: If improvement is not possible, the solution reached is optimal.

2. Standard Typical Linear Program Formula in the Simplex Method

This means converting the mathematical model of the problem under study into a standard form, as it must:

- ✓ All program constraints must be in the form of equations using dummy variables in the objective function and model constraints, as shown in the table:
- ✓ All variables satisfy the conditions of non-negativity;
- ✓ an objective function that seeks to achieve idealization (maximization or minimization);
- ✓ The right side of the constraint (constant values b_i) must be positive or non-existent.

3. Simplex method according to the Canonical formula in case of maximization

- Restrictions of a type less than or equal to ≤: This restriction means that the left side of the inequality is less than the right side and achieving equality between the two parties is only done by adding a variable with a positive signal to the left side equal to the difference between the two sides, called the imaginary variable, which we code as S_i .

constraint signe	The mechanism of using slack variables in the constraint	Mechanism for Using S_i Variables in the Objective Function Maximize
is less than or equal to	$+1S_i$	$+0S_i$

In order for all variables to be represented in all equations, we add variables of differences with a coefficient of zero in equations in which one of these variables does not appear.

-The economic meaning of the variable S_i in a state smaller than or equal to:

The variable S_i represents the unused quantity of the available resource b_i so this variable is called: **slack variable.**

Example: Solving the following linear program using the simplex method

$$\text{Max}: Z = 2X_1 + X_2$$

$$\text{S/T} \begin{cases} 3X_1 + X_2 \leq 15 \\ X_1 + 2X_2 \leq 12 \\ X_1 \geq 0,\ X_2 \geq 0 \end{cases}$$

1/ Converting the linear program to the standard (typical) form:

-The first constraint: $3X_1+X_2\leq15$, we add a Slack variable and let it be S_1, the constraint becomes an equal of the figure:

$3X_1+X_2+S_1=15$.

-The second constraint: $X_1+2X_2\leq12$, we add a slack variable and let it be S_2, the constraint becomes an equal of the figure:

$X_1+2X_2+S_2=12$.

$3X_1+X_2+1S_1+0S_2=15$
$X_1+2X_2+0S_1+1S_2=12$

Since the number of entries is 2, the square unit matrix of rank (2x2) must be obtained

- **The objective function in standard form:** When adding gap variables to constraints, they must also be added to the objective function with zero coefficients as follows:

$$\text{Max}: Z=2X_1+X_2+0S_1+0S_2$$

-Since the variables of the difference are non-negative variables, this must be written in the standard format, and the final model format of the program is given as follows:

$$\text{Max}: Z=2X_1+X_2+0S_1+0S_2$$

$$\begin{cases} 3X_1+X_2+1S_1+0S_2=15 \\ X_1+2X_2+0S_1+1S_2=12 \end{cases}$$

$$X_1\geq0,\ X_2\geq0,\ S_1\geq0,\ S_2\geq0$$

2. Initial Simplex Table

C_j		2	1	0	0	B_i (solution)	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2		
0	S_1	3	1	1	0	15	15/3= 5
0	S_2	1	2	0	1	12	12/1= 12
$Z=\sum_{j=1}^{n} C_j X_j$		0	0	0	0	Z=0	
C_j-Z_j		2	1	0	0		

Outgoing row

Incoming column

Pivot element

Identity matrix

- **Basic Variables:** S_1, S_2
- **Non-Basic Variables:** X_1, X_2

It is noted from the first basic solution table that the variables within the base are the variables that form the unity matrix, while the variables outside the base are the structural variables that we are looking for the optimal values that achieve the maximum function.

- The solution highlighted in the first table is a preliminary solution, and this solution does not make any profit, which means that no unit of products X_1 and X_2 is produced, this solution is a corner point of the potential area.

-In the first basic solution table, in the case of maximization, we always find that the value of the objective function Z=0, and the values of the variables $X_1=0$ and $X_2=0$.

3. Ideal solution test:

Before forming the second basic solution table, the optimization of the solution must be tested, and this is reflected in the search for whether there is a model variable that exists outside the base variables that can improve the value of the objective function (towards an increase in the case of maximization and towards a decrease in the case of minimization).
The optimization of the solution is tested in the case of maximization through the last line **C_j-Z_j**. If all the values are negative or equal to zero, the solution is ideal, but if one or some of the values is positive, the solution is not optimal and requires improvement.
Returning to the example, we find that the values of the last line are some positive and some equal to zero, so the solution is not optimal and requires improvement.

4. Solution optimization: To optimize the solution we follow the steps below

Step 1: Determine the largest positive value in the last line **C_j-Z_j** of the simplex table, and the column that includes this value is called the **Incoming column**. It is clear from our example that the largest positive value is 2 and therefore the incoming column is the column that contains this value.

Step 2: To determine the **Outgoing row** we calculate the ratio **B_i/a_{ij}**, which means dividing each digit of column **B_i** by the **positive** number in the pivot column that is in its row. The element in the pivot shaft that results in the smallest ratio is called the **pivot element.** If more than one number leads to the same percentage, one of them is chosen, and if there is no positive number in the pillar, the program has no solution.
Returning to our example, we find that the smallest ratio is 5 and therefore the **pivot element is the number 3.**

Step 3: Take the pivot element number 1, and the rest of the outgoing row elements are divided by the pivot element (3), while the incoming column elements take the value 0. The rest of the other items are calculated as follows:

Pivot element →

a	b
d	c

The value of the new element c is calculated according to the following rule:

$$C' = \frac{(a*c)-(b*d)}{a}$$

Step 4: Replace the variable that will come out of the foundation (which is in the outgoing row) with the variable that will enter the foundation (which is in the pivot column).
Going back to our example, the variable that will come out of the base is S_1, to be replaced by the variable X_1.
From the previous steps, the second basic solution table can be formed as follows:

Second Simplex Table

C_j		2	1	0	0	B_i	B_i/a_{ij}
C_b	X_j	X_1	X_2	S_1	S_2	(Solution)	
2	X_1	1	1/3	1/3	0	5	15
0	S_2	0	5/3	-1/3	1	7	21/5
$Z=\sum_{j=1}^{n} C_j X_j$		2	2/3	2/3	0	Z=10	
C_j-Z_j		0	1/3	-2/3	0		

Outgoing row (the S_2 row); Incoming column (X_2); Pivot element (5/3)

We note from the table that the value of the objective function has improved from 0 to 10.
- Basic Variables: X_1= 5, S_2= 7
-Non- Basic Variables: X_2=0, S_1=0

To be sure, we substitute in the objective function: Z=2(5) +0 +0(0) +0(7) = 10

- It is also noted that the values of the last line include one positive value, which indicates that the solution is not optimal and can be improved by following the same steps as before.
- The largest positive value in the last line is 3/1, so the pivot column includes this value.
- By calculating the ratio B_i/a_{ij}, we find that the smallest ratio is 5/21 and therefore the pivot element is the number **5/3.**

Applying the rest of the previous steps, the third basic solution table is formed as follows:

Optimal Simplex Table

C_b →		**2**	**1**	**0**	**0**	B_i	B_i/a_{ij}
↓		X_1	X_2	S_1	S_2		
2	X_1	1	0	6/15	-1/5	18/5	
1	X_2	0	1	-1/5	3/5	21/5	
$Z=\sum_{j=1}^{n} C_j X_j$		**2**	**1**	**9/15**	**1/5**	**Z=(2*18/5)+(1*21/5)=57/5**	
C_j-Z_j		**0**	**0**	**-9/15**	**-1/5**		

It is clear from the values of the last line that some values are negative and some are equal to zero and therefore the optimal solution has been reached and the third basic table is the optimal table. Whereas, the values of the variables:

-Basic Variables: X_1= 18/5, X_2= 21/5

- Non- basic Variables: S_2=0, S_1=0

To be sure, we substitute in the objective function: Z=2(18/5) +21/5+0(0) + 0(0) = 57/5

<u>Simplex method according to the canonical form in case of minimizing</u>

There are two ways to solve the linear model in the state of minimize:

The first method: We work to turn the problem of linear programming into a maximization problem, by multiplying the two ends of the goal function by -1, so that the objective function becomes as follows:

Min Z= Max (-Z)

Minimizing Z means completely maximizing its negative value (-Z), the rest of the solution steps are applied exactly as in the previous example in the case of maximization.

2 The second method

According to the canonical formula of the linear program in the case of inferiority, all the constraints are of the form greater or equal, and to convert the linear program to the standard formula for the application of the simplex method, **the surplus variable** is subtracted from the left side of the constraint and is called the gap variable and is symbolized by S_i, and since the coefficient of the gap variable is negative, which does not allow us to obtain the unity matrix in the constraint coefficients, so artificial variables are added to the left side of the constraint, symbolized by A_i. The gap variable is shown with a coefficient of zero in the objective function, the artificial variable is shown with a coefficient of M in the objective function, which symbolizes a very large numerical coefficient, **and the coefficient signal is negative in the case of maximization and positive in the case of minimization**.

The following table illustrates this:

constraint signe	**The mechanism of using surplus variables in the constraint**	**The mechanism of using Si variables in the objective function**
is greater than or equal to	$-1S_i+1A_i$	$+0S_i+MA_i$

Example: Solving the following linear program using the simplex method

Min : $Z=5X_1+7X_2$

S / T
$X_1- X_2\geq 3$
$X_1+2X_2\geq 6$
$X_1\geq 0,\ X_2\geq 0$

Solution:

1. Converting the program to the standard form:

Min: $Z=5X_1+7X_2+0S_1+0S_2$
$X_1-X_2-1S_1+0S_2=3$
$X_1+2X_2+0S_1-1S_2=6$
$X_1\geq 0,\ X_2\geq 0,\ S_1\geq 0,\ S_2\geq 0$

2. Obtaining the Initial Solution:

This program is effective in its typical form, but it does not enable us to obtain an initial solution that is permissible. Whereas, giving the original variables X_1, X_2 a value equal to zero gives us the following solution:

$X_1=0,\ X_2=0,\ S_1= -30,\ S_2 = -40,\ Z=0$

This solution does not meet the conditions of non-negativity, as the surplus variables are negative, (they cannot be part of the unit matrix that contains the basic variables). In this case, **artificial variables** must be added to the constraints as well as the objective function according to the previous rule listed in the table.

Min: $Z=5X_1+7X_2+0S_1+0S_2+MA_1+MA_2$
$X_1-X_2-1S_1+0S_2+1A_1+0A_2=3$
$X_1+2X_2+0S_1-1S_2+0A_1+1A_2=6$
$X_1\geq 0,\ X_2\geq 0,\ S_1\geq 0,\ S_2\geq 0,\ A_1\geq 0,\ A_2\geq 0$

- **Decision variables:** $x_1=0,\ x_2=0$
- **Surplus variables:** $S_1=0,\ S_2=0$
- **Artificial variants:** $A_2= 6,\ A_1= 3$

- **Objective function value**: Min: $Z=5(0) +7(0) +0S_1+0S_2 +3M_1 +6M_2$

Min: $Z=9M$

It is noted that the above solution contains very large values in the objective function, and this is due to the emergence of artificial variables with zero certainty in the solution. Let's list the first simplex table.

Initial Simplex Table

C_b		5	7	0	0	M	M	B_i	B_i/a_{ij}
		X_1	X_2	S_1	S_2	A_1	A_2		
M	A_1	1	-1	-1	0	1	0	3	3/1= 3
M	A_2	1	2	0	-1	0	1	6	6/1= 6
$Z_j=\sum_{j=1}^{n} C_jX_j$		2M	M	-M	-M	M	M	Z=9M	
C_j-Z_j		5-2M	7-M	M	M	0	0		

Pivot element

- Basic variables: A_1, A_2
- Non-basic variables: X_1, X_2, S_1, S_2
-Step 3: Evaluate the possibility of improving the solution
Referring to the values of the last line, we note that there are two negative values and this means that the optimal solution has not yet been reached.

Second Simplex Table

C_b		5	7	0	0	M	M	B_i	B_i/a_{ij}
		X_1	X_2	S_1	S_2	A_1	A_2		
5	X_1	1	-1	-1	0	1	0	3	/
M	A_2	0	3	1	-1	-1	1	3	3/3= 1
$Z_j= \sum_{j=1}^{n} C_j X_j$		5	-5+3M	-5+M	-M	5-M	M	Z=15+3M	
C_j-Z_j		0	12-3M	5-M	M	5	0		

Pivot element

- Incoming variable is: X_2 and outgoing variable is: A_2

Optimal Simplex Table

C_b		5	7	0	0	M	M	B_i	B_i/a_{ij}
		X_1	X_2	S_1	S_2	A_1	A_2		
5	X_1	1	0	-2/3	-1/3	2/3	1/3	4	
7	X_2	0	1	1/3	-1/3	-1/3	1/3	1	
$Z_j= \sum_{j=1}^{n} C_j X_j$		5	7	-1	-4	1	4	Z=27	
C_j-Z_j		0	0	1	4	M-1	M-4		

It is clear from the values of the last line that some values are positive and some are equal to zero and therefore the optimal solution has been reached and the third table is the optimal table. Whereas, the values of the variables:
- Basic Variables: $X_1= 4$, $X_2= 1$
- Non-basic variables: $A_2=0$, $A_1=0$, $S_2=0$, $S_1=0$
To be sure, we substitute in the objective function: Z=5(4) +7 (1)=27