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Chapter 2: Linear Programming

Linear programming is a graphical and mathematical tool that is concerned with building mathematical models of a problem in one of the following ways: graphical method, simplex method, transfer method...etc.)

1. Definition of linear programming:

- Linear programming is defined as a mathematical method of distributing a set of constraints and fixed factors so that this distribution achieves the best possible result, that is, that the distribution is perfect.
- Linear programming is also defined as a mathematical method or technique that seeks a solution or solutions to an economic problem, whether (productivity, financial, transfer, project analysis...etc.) and choose the best solution that represents the optimal solution.

Thus, the goal of linear programming is to find the optimal use of finite resources, by means of a mathematical function composed of a set of constraints in the form of equations, inequalities, or both of the first order.

2. Requirements for building linear programming models

A linear program is nothing but a mathematical program that has distinctive characteristics. Any linear program is required to achieve a set of characteristics, which are summarized as follows:

- The existence of a specific goal for which we seek to achieve the best level of performance, expressed in written form. The goal is formulated as a numerical function of several variables.
- Availability of a range of options (possible solutions), as the existence of a single option (a single solution) does not require the use of any form to search for it.
- **Limited resources:** The need for linear programming lies in the optimal distribution of these resources, as if resources were available in unlimited quantities, there would be no need to use linear programming or others.
- All relationships are linear whether in the goal function or in constraints, i.e. direct changes between the values of the function and the values of the variables, also resources are consumed linearly.
- All decision variables are non-negative (greater or equal to zero), as the variables express economic amounts of resources, products, or others, and therefore cannot be negative.

3. Linear programming hypotheses: Linear programming is adopted from a number of assumptions, the most important of which are:

- **A. Proportionality:** Whether for the objective function or the constraints. For example, if we need three units of raw materials to produce one complete unit of a particular product, then we need thirty units of raw materials to produce ten units of this product.
- **B. Additivity:** This means that if the profit from the first product is equal to 10 dinars and the profit from the second product is equal to 20 dinars, and one unit of the first product and one unit of the second product are produced, then the total profit resulting from producing and selling one unit of each product will be 10+20= 30 dinars**.**
- **C. Divisibility (continuity):** What is meant here is that the solution to the linear programming problem is not necessarily in whole numbers (natural), which means accepting fractions as values for decision factors.
- **Linearity:** Where the objective function and constraints are required to be equations or inequalities containing blindness of first-order variables.

- **Non-negativity:** This means that all the values of the decision variables must be positive (non-negative). Negative values of physical quantities are impossible, for example, we cannot produce a negative number of chairs or T-shirts.
- **Certainty:** All information on which linear programming depends must be certain and does not change during the study period, whether the matter is deepened by the objective function or the constraints objective function or the constraints.

4. Linear Programming Usage Areas:

Linear programming is used in all economic matters aimed at searching for the values of economic variables with the aim of finding optimal use.

Minimization Cases

Cost minimization;
Minimize losses;
Low headcount;
Low gross wages.

Maximizations cases

Maximizing profits;
Maximizing production;
Maximize storage capacities;
Maximizing the use of capital
Maximizing laborforce

5. Constructing (Formulating) the Mathematical Model of Linear Programming:

We mean to transform the problem from a listed reality in the form of literary expressions into a problem formulated in a clear mathematical form. To build the mathematical model, the linear program can be followed as follows:

- **Determining variables:** It is considered the first step in building the mathematical model of the linear program, as the variables to be maximized in the case of maximization or to be minimized in the case of minimizing must be determined, as these variables enter into the writing of the mathematical form in all subsequent steps.
- **Forming the problem table:** After determining the variables, we should form the problem table so that this table contains all the elements of the problem from the variables, constraints and quantities specified for the objective function.
- **Determining the objective function:** It is the possibility of expressing the desired objective that we want to achieve in the form of a linear function and obtaining a numerical value for it. We seek to find the maximum end of it if the desired goal is to maximize profit, or to find the minimum end of it if the goal is to minimize cost. The goal function consists of the variables that we identified in the first step multiplied by a special coefficient that expresses the profit of one unit in the case of maximization, or expresses the cost of one unit in the case of minimization.

  Mathematically, it is expressed as follows:

Maximizing case: $\text{Max: } Z=\sum_{j=1}^{n} C_j X_j$

Minimizing case: $\text{Min: } Z=\sum_{j=1}^{n} C_j X_j$

- **Determining constraints:** It is the possibility of expressing the relationship between the variables and the available possibilities in a linear form that shows what each production unit needs from each resource of the limited available resources in the form of inequalities, equations, or both.
  Mathematically, it is expressed as follows:

$$\sum_{j=1}^{n} a_{ij} X_j \ (\leq, \geq, =) b_i, (i = 1,2 \dots, m)$$

- **Non-negative condition:** The variables forming the linear program must be non-negative, i.e. positive or equal to zero.
  Mathematically, it is expressed as follows:

$$X_j \geq 0, (j=1, 2, \dots, n)$$

Where: Z: represents the objective function to be maximized or minimized
Cj: Decision variable coefficient representing profit margin, unit cost, sale price...etc.
Xj: Decision variable no. j.
a_{ij}: the technical coefficients of the model.
b_i: represents the limited resources of type i

Example: A small-scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in the machine shop and 2 hours of work in the assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of dinars 3 per unit of product P and dinars. 4 per unit of product Q. Give a mathematical formulation of the problem as to maximise profit.
Solution:
let X_1 and X_2 be the number of units of product P and Q, which are to be produced. Here x and y are the decision variables. Suppose Z is the profit function. Since one unit of product P and one unit of product Q gives the profit of rupees 3 and rupees 4, respectively, the objective function is Maximize

$$\mathbf{Z = 3X_1 + 4X_2}$$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine shop	Assembly shop	Profit
Product P	**2 hours**	**4 hours**	**3 dinars per unit**
Product Q	**3 hours**	**2 hours**	**4 dinars per unit**
Available hours per day	**16 hours**	**22 hours**	

Total hours of machine shop required for both types of product = $\mathbf{2X_1 + 3X_2}$

Total hours of assembly shop required for both types of product =**$4X_1 + 2X_2$**
Hence, the constraints as per the limited available resources are:
$2X_1 + 3X_2 \leq 16$ and $4X_1 + 2X_2 \leq 22$
Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are: $X_1 \geq 0$, $X_2 \geq 0$
Thus, the mathematical formulation of the given problem is:
Maximise $Z = 3X_1 + 4X_2$
Subject to the constraints
$2X_1 + 3X_2 \leq 16$
$4X_1 + 2X_2 \leq 22$
And non-negative restrictions
$X_1 \geq 0$, $X_2 \geq 0$

6. Formulate linear models: Linear programming problems can be written according to three formulas:

- **Canonical form of LP** :All entries are written with the same sign either ≤ or ≥ only.
  - If the formula has a less than or equal sign, the objective function is in the form of Max;
  - If the formula has a greater or equal sign, then the objective function is in the form of Min.
  - All decision changes are non-negative, and all restrictions are disparities.
  - All decision variables are non-negative.
  - b_i Signal undefined.
- **Standard form:** It is the form in which all constraints in the form of a signal are equal to (=) only (excluding the constraints of non-negativity), and the objective function is either in the form of maximization or minimization.
  - All decision variables are non-negative.
  - b_i is non-negative.
- **Mixed form:** A linear programming problem is said to be of mixed format if it does not take the legal or standard format. The constraints in the linear program are written in a mixed form containing all the signs (≤, =, ≥) of the DF function, which is in the form of Max maximization or in the form of Min minimization.

-Converting mixed format to legal or standard format:
It is possible to move from the mixed formula to the regular or standard formula using some initial transformations in line with the formula to be reached.
Shunt 1: Minimization (Minimized) of the objective function can be converted to Maximized (Maximized) and vice versa by multiplying the objective function by (-1).
From: **min Z= Max-Z**
Example: Max: $Z=2X_1+3X_2$ Min $Z= -2X_1-3X_2$

Shunt 2: Any differential with a certain signal can be replaced by a differential from an opposite signal by multiplying its two ends by (-1).

$$ax \geq b \quad -ax \leq -b$$

$$3X_1+2X_2 \geq 12 \quad -3X_1-2X_2 \leq -12$$

Shunt 3: Each equal can be replaced by two opposite inequalities

$$ax + b = 0 \quad \begin{cases} ax \geq b \\ ax \leq b \end{cases}$$

Example:

$$5X1 + 2X2 = 6 \rightarrow \begin{cases} 5X1 + 2X2 \geq 6 \\ 5X1 + 2X2 \leq 6 \end{cases}$$

Shunt 4: Each left-end inequality forms an absolute value that can be replaced by two regular inequalities if p and q are greater than or equal to zero then:

$$|a.x| \leq p \rightarrow \begin{cases} a.x \leq p \\ a.x \geq -p \end{cases} \quad , \quad |a.x| \geq q \rightarrow \begin{cases} a.x \geq q \\ a.x \leq -q \end{cases}$$

Shunt 5: Each inequality can be transformed into equality:

- Adding a new non-negative duster to its left end in a **less or equal** condition and this variable is called the difference variable;
- By subtracting a new non-negative variable from its left side in a **large or equal** condition, this variable is called an auxiliary variable.

Shunt 6: Each non-specific variable (positive, negative, or zero value) can be replaced by a differential of two non-negative variables. For example, if X is undefined, the signal is replaced by: $X^+ - X^-$, where: $X^+ \geq 0$, $X^- \geq 0$.

Example: we have the following linear program**:**

$$\text{Min: } Z = 3X_1 - 3X_2 + 7X_3$$
$$X_1 + X_2 + 3X_3 \leq 40$$
$$X_1 + 9X_2 - 7X_3 \geq 50$$
$$5X1 + 3X2 = 20$$
$$[5X_1 + 8X_2] \leq 100$$
$$X_3\text{: Unrestricted variable}$$

-Writing the program according to the Canonical and standard forms based on previous transfers:

Solution: Canonical form

$$\text{Max : } Z = -3X_1 + 3X_2 - 7(X_3^+ - X_3^-)$$
$$X_1 + X_2 + 3(X_3^+ - X_3^-) \leq 40$$
$$-X_1 - 9X_2 + 7(X_3^+ - X_3^-) \leq -50$$
$$5X_1 + 3X_2 \leq 20$$
$$-5X_1 - 3X_2 \leq -20$$
$$5X_1 + 8X_2 \leq 100$$
$$-5X_1 - 8X_2 \leq 100$$
$$X_1 \geq 0,\ X_2 \geq 0,\ X_3^+ \geq 0,\ X_3^- \geq 0$$

Standard Form

$$\text{Max}: Z=-3X_1+3X_2-7(X_3^+-X_3^-)+0S_1+0S_2+0S_3$$

$$X_1+X_2+3(X_3^+-X_3^-)+1S_1=40$$

$$-X_1-9X_2+7(X_3^+-X_3^-)-S_2=50$$

$$5X_1+3X_2=20$$

$$5X_1+8X_2+1S_3=100$$

$$-5X_1-8X_2+1S_4=100$$

$$X_1\geq 0,\ X_2\geq 0,\ X_3^+\geq 0,\ X_3^-\geq 0,\ S_1\geq 0,\ S_2\geq 0,\ S_3\geq 0$$