# Contents

# Chapter 1

# Single and multiple integrals

## 1.1 Riemann Integral

The Riemann integral of a function is a method of assigning a number to the area under the curve of a function over a certain interval.

### 1.1.1 Subdivisions

**Definition 1.1** *A subdivision (or partition) $P$ of the interval $[a, b]$ is a finite sequence of points:*

$$P = \{x_0, x_1, \dots, x_n\} \quad \textit{where} \quad a = x_0 < x_1 < \cdots < x_n = b, \quad n \in \mathbb{N}.$$

**Definition 1.2** *A subdivision $P'$, is said to be finer than another subdivision $P$ if $P$ means that every point of $P'$ is also a point of $P$.*

**Definition 1.3** *A uniform subdivision of $[a, b]$ is a subdivision where each subinterval $\Delta x_i$ has the same length:*

$$\Delta x = \frac{b-a}{n} \quad \textit{for all}\, i = 1, 2, \dots, n$$

### 1.1.2 Darboux Sums

**Definition 1.4** *A function $f : [a, b] \to \mathbb{R}$ is said to be bounded if there exists a constant $M > 0$ such that for all $x \in [a, b]$, we have:*

$$\sup |f(x)| \leq +\infty$$

Given a bounded function $f$ on $[a, b]$ and a partition $P = \{x_0, x_1, \ldots, x_n\}$, we define the Darboux sums (upper and lower) as follows:

**Definition 1.5** *The lower Darboux sum $L(f, P)$ is given by:*

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

*where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ is the infimum of $f$ on the subinterval $[x_{i-1}, x_i]$.*

**Definition 1.6** *The upper Darboux sum $U(f, P)$ is defined as:*

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

*where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ is the supremum of $f$ in the subinterval $[x_{i-1}, x_i]$.*

### 1.1.3 Riemann Sums

we consider the Riemann sum, defined as:

$$s_n = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_{i-1})$$

and

$$S_n = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i)$$

More generally, if $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfies $\varepsilon_i \in [x_{i-1}, x_i]$. The associated Riemann sum is given by:

$$S(f, P, \varepsilon) = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(\varepsilon_i)$$

**Definition 1.7** *We say that $f$ is Riemann integrable on $[a, b]$ if and only if:*

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$$

*exists and is finite. In that case, we write:*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$$

***Theorem 1.1*** *If $f$ is a continuous function on the interval $[a, b]$, then the Riemann sums $(S_n)$ and $(s_n)$ converge to the Riemann integral $\int_a^b f(x)\,dx$.*

***Remark 1.1*** *If $P$ is a uniform subdivision, we have:*

$$s_n = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} i\right)$$

*and*

$$S_n = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)$$

***Example 1.1*** *Compute the following sum:*

$$S = \sum_{k=1}^{\infty} \frac{k}{n^2 + k}$$

*First, rewrite the terms to resemble a Riemann sum form:*

$$S_n = \sum_{k=1}^{n} \frac{k}{n^2 + k^2} = \sum_{k=1}^{n} \frac{k}{n^2\left(1 + \frac{k^2}{n^2}\right)} = \frac{1}{n} \sum_{k=1}^{n} \frac{k/n}{1 + \left(\frac{k}{n}\right)^2}$$

*Next, express this sum as a Riemann sum:*

$$S_n = \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f\left(a + \frac{b-a}{n} k\right) = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k}{n}\right)$$

*Now, let's choose $a = 0$, $b = 1$, and $f(x) = \frac{x}{1+x^2}$. Therefore:*

$$S_n = \frac{1}{n} \sum_{k=1}^{n} \frac{k/n}{1 + (k/n)^2}$$

*Since the function $f(x) = \frac{x}{1+x^2}$ is continuous on the interval $[0,1]$, the Riemann sum $S_n$ converges as $n \to \infty$, and we have the following limit:*

$$\lim_{n\to\infty} S_n = \int_0^1 \frac{x}{1+x^2}\,dx$$

*Now, compute the integral:*

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\left(\ln(1+x^2)\right)\Big|_0^1 = \frac{1}{2}\ln(2)$$

*Thus, the sum converges to:*

$$\lim_{n\to\infty} S_n = \frac{\ln(2)}{2}$$

## 1.2 Antiderivative of a Continuous Function

*Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a continuous function.*

***Definition 1.8*** *A function $F : D \to \mathbb{R}$ is called an antiderivative of $f$ in $D$ if and only if:*

- *$F$ is differentiable on $D$,*
- *$F' = f$ on $D$.*

***Theorem 1.2*** *Any continuous function $f : [a,b] \to \mathbb{R}$ possesses an antiderivative. We write:*

$$F(x) = \int_a^x f(t)\,dt$$

*and*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

***Example 1.2*** *Find the antiderivative of $f(x) = 3x^2$.*

$$F(x) = \int (3x^2 + e^{-x} - \cos 2x)\,dx = x^3 - e^{-x} + 2\sin 2x + C.$$

*Thus, $F'(x) = 3x^2 + e^{-x} - \cos 2x$, and the function $F(x) = x^3 - e^{-x} + 2\sin 2x$ is an antiderivative of $f(x)$.*

# 1.3 Double Integrals

***Definition 1.9*** *The double integral of the function $f(x,y)$ over the domain $D$ is defined as the limit of the sequence $(v_{n_t})_{n_t \in \mathbb{R}}$, as the largest subregion $\Delta s_k \to 0$ and $n \to \infty$. We denote this limit as:*

$$\lim_{\substack{\max \Delta s_k \to 0 \\ n \to \infty}} \sum_{k=1}^{n} f(p_k)\Delta s_k = \iint_D f(x,y)\,dx\,dy$$

## 1.3.1 Properties of Double Integrals

*Double integrals have several important properties, which are similar to those of single integrals:*

*Let $f(x,y)$ be a continuous function, and $D = D_1 \cup D_2 \subset \mathbb{R}^2$ such that $D_1 \cap D_2 = \emptyset$*

1. ***Linearity:*** *If $f(x,y)$ and $g(x,y)$ are integrable over the region $D$, and $\alpha, \beta \in \mathbb{R}$, then:*

$$\int\int_D [\alpha f(x,y) + \beta g(x,y)]\,dA = \alpha \int\int_D f(x,y)\,dA + \beta \int\int_D g(x,y)\,dA$$

2. ***Additivity:*** *If $D_1$ and $D_2$ are two non-overlapping regions within $D$, then:*

$$\int\int_D f(x,y)\,dA = \int\int_{D_1} f(x,y)\,dA + \int\int_{D_2} f(x,y)\,dA$$

3. ***Positivity:*** *If $f(x,y) \geq 0$ for all $(x,y) \in D$, then:*

$$\int\int_D f(x,y)\,dA \geq 0$$

4. ***Comparison Property:*** *If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then:*

$$\int\int_D f(x,y)\,dA \leq \int\int_D f(x,y)\,dA$$

## 1.3.2 Double Integral Calculations

*Let us consider a domain $D$ in the xy-plane, which is bounded by two curves $y = \varphi(x)$ and $y = \psi(x)$, where $\varphi(x) \geq \psi(x)$, as well as the vertical lines $x = a$ and $x = b$.*

***Definition 1.10*** *The domain $D$ is said to be regular with respect to the $x$ axis (or the $y$ axis) if any line parallel to the $x$-axis (or the $y$-axis) passing through a point in $D$ intersects its boundary at exactly two points, $M_1$ and $M_2$ (or $N_1$ and $N_2$).*

***Remark 1.2*** *If the domain $D$ is regular with respect to both the $x$-axis and the $y$-axis, it is simply called a regular domain.*

***Theorem 1.3*** *Let $D \subset \mathbb{R}^2$ be a regular domain and $f : D \to \mathbb{R}$ a continuous function on $D$. Then:*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left( \int_{\psi(x)}^{\varphi(x)} f(x,y)\,dy \right) dx$$

*where:*

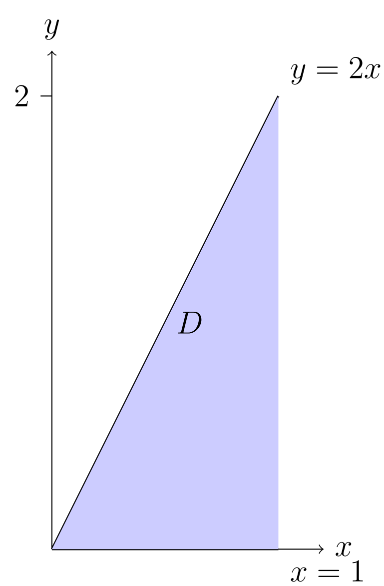
- *$[a,b]$ is the orthogonal projection of $D$ onto the $x$-axis.*
- *$[\psi(x), \varphi(x)]$ is the intersection of $D$ with the line where $x$ is constant.*

***Example 1.3*** *Calculate the following integral*

$$I = \iint_D x\,dy\,dx$$

*where $D$ is the region defined by $0 \leq x \leq 1$ and $0 \leq y \leq 2x$.*

*The region $D$ is bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 2x$. The area is shown in the figure below.*



*We calculate the double integral as follows:*

$$I = \int_0^1 \int_0^{2x} x \, dy \, dx$$

*Step 1: Integrating with respect to y:*

$$\int_0^{2x} x \, dy = 2x^2$$

*Step 2: Integrating with respect to x:*

$$I = \int_0^1 2x^2 \, dx = \frac{2}{3}$$

*Thus, the value of the integral is:*

$$I = \frac{2}{3}$$

***Theorem 1.4*** *Let $f$ be a continuous function over a rectangle $D = [a, b] \times [c, d]$. Then we have:*

$$\iint_D f(x, y) \, dx \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy$$

***Example 1.4*** *Consider the domain $D = [1, 2] \times [0, 2] \subset \mathbb{R}^2$ and the function $f : D \to \mathbb{R}$ defined as $f(x, y) = ye^{xy}$. We want to compute:*

$$I = \iint_D f(x, y) \, dx \, dy$$

*Using Fubini's theorem, we proceed as follows:*

$$I = \int_0^2 \left( \int_1^2 ye^{xy} \, dx \right) dy$$

*First, compute the inner integral:*

$$\int_1^2 e^{xy} \, dx = \frac{1}{y} \left( e^{2y} - e^y \right)$$

*Now, substitute this into the outer integral:*

$$I = \int_0^2 \frac{1}{y} \left( e^{2y} - e^y \right) y \, dy = \int_0^2 \left( e^{2y} - e^y \right) \, dy$$

*Next, compute the final integral:*

$$I = \left[ \frac{1}{2} e^{2y} - e^y \right]_0^2 = \frac{1}{2} e^4 - e^2 + 1 - (\frac{1}{2} e^0 - e^0) = \frac{1}{2} e^4 - e^2 + \frac{1}{2}$$

***Corollary 1.1*** *Let $f_1(x)$ and $f_2(y)$ be two functions, then:*

$$\iint_D f_1(x) f_2(y) \, dx \, dy = \left( \int_a^b f_1(x) \, dx \right) \left( \int_c^d f_2(y) \, dy \right)$$

## Double Integration with a Change of Variables

*Consider a continuous function on a domain $D \subset \mathbb{R}^2$. Using a change of variables, the double integral can be transformed accordingly. The method depends on the transformation used, such as polar, cylindrical, or spherical coordinates.*

## General Case: Change of Variables

*Suppose we perform the following change of variables:*

$$\begin{cases} x = \varphi(u, v) \\ y = \psi(u, v) \end{cases}$$

*When $(x, y)$ varies over the domain $D$, $(u, v)$ varies over the domain $D_1$. The double integral can be rewritten as:*

$$\iint_D f(x, y)\, dx\, dy = \iint_{D_1} f(\varphi(u, v), \psi(u, v))\, |\det(J)|\, du\, dv$$

*where $|\det(J)|$ is the absolute value of the Jacobian determinant:*

$$\det(J) = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} \neq 0$$

***Example 1.5*** *Consider the integral:*

$$I = \iint_D (x + y)\, dx\, dy$$

*where $D = \{(x, y) \in \mathbb{R}^2 : 1 \leqslant x.y \leqslant 2, -1 \leqslant x + 3y \leqslant 1\}$.*

*We make a change of variables to the domain $D_1 = \{(u, v) \in \mathbb{R}^2 : 1 \leqslant u \leqslant 2, -1 \leqslant v \leqslant 1\}$. Applying the transformation, we get:*

$$I = \frac{1}{16} \iint_{D_1} (2u + 2v)\, du\, dv$$

*Now, compute the integral:*

$$I = \frac{1}{8} \int_{-1}^{1} \int_{1}^{2} (u + v)\, du\, dv$$

*First, compute the inner integral:*

$$\int_1^2 (u + v)\, du = \left[\frac{1}{2}u^2 + vu\right]_1^2 = \frac{1}{2}(2^2 - 1^2) + v(2 - 1) = \frac{3}{2} + v$$

*Substituting into the outer integral:*

$$I = \frac{1}{8} \int_{-1}^{1} \left( \frac{3}{2} + v \right) \, dv$$

*Now, compute the final integral:*

$$I = \frac{1}{8} \left[ \frac{3}{2}v + \frac{1}{2}v^2 \right]_{-1}^{1} = \frac{1}{8} \left[ \frac{3}{2}(1 - (-1)) + \frac{1}{2}(1^2 - (-1)^2) \right]$$

*Thus, the value of the integral is:*

$$I = \frac{3}{8}$$

## Change to Polar Coordinates

*In this case, let $u = \theta$ and $v = r$. The change of variables is given by:*

$$\begin{cases} x = r\cos\theta = \varphi(r, \theta) \\ y = r\sin\theta = \psi(r, \theta) \end{cases}$$

*where $(u, v)$ varies in the domain $D_1$, the double integral over $D$ can be expressed as:*

$$\iint_D f(x, y) \, dx \, dy = \iint_{D_1} f(\varphi(u, v), \psi(u, v)) \, |J| \, du \, dv,$$

*where $J$ is the Jacobian determinant of the transformation.*

***Example 1.6*** *Calculate the integral*

$$I = \iint_D \frac{1}{1 + x^2 + y^2} \, dx \, dy,$$

*where*

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \right\}.$$

*To solve this, we use polar coordinates $(r, \theta)$ where:*

$$x = r\cos\theta, \quad y = r\sin\theta.$$

*The domain $D$ in polar coordinates is:*

$$D_1 = \{ (r, \theta) \mid 0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi \}.$$

*The Jacobian determinant of the transformation from Cartesian to polar coordinates is:*

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

*Thus, the integral becomes:*

$$\iint_D \frac{1}{1+x^2+y^2}\,dx\,dy = \iint_{D_1} \frac{r\,dr\,d\theta}{1+r^2}.$$

*Compute the integral:*

$$\iint_{D_1} \frac{r\,dr\,d\theta}{1+r^2} = \int_0^{2\pi} \left( \int_0^1 \frac{r\,dr}{1+r^2} \right) d\theta = \int_0^{2\pi} \left[ \frac{1}{2}\ln(1+r^2) \right]_0^1 d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} \ln\left( \frac{1+1^2}{1+0^2} \right) d\theta = \int_0^{2\pi} \frac{1}{2}\ln 2\,d\theta = \frac{1}{2}\ln 2 \cdot [\theta]_0^{2\pi} = \frac{1}{2}\ln 2 \cdot 2\pi$$
$$= \pi \ln 2.$$

### 1.3.3 Applications of Double Integrals

## 1. Area calculation

*The area of the domain $D$ is given by the formula:*

$$A(D) = \iint_D dx\,dy.$$

***Example 1.7*** *Calculate the area of $D$ where:*

$$D = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < y < 1 \text{ and } 0 < x < e^y \right\}.$$

*Calculate the area as follows:*

$$\begin{aligned} A(D) &= \iint_D dx\,dy \\ &= \int_0^1 \int_0^{e^y} dx\,dy \\ &= \int_0^1 e^y\,dy \\ &= e^1 - e^0 \\ &= e - 1. \end{aligned}$$

## 2. Mass calculation

*Let $\rho(x, y)$ be the density of the material within the domain $D$. The total quantity of the material in the domain is given by:*

$$m = \iint_D \rho(x, y)\, dx\, dy.$$

## 3. Surface area calculation

*The area of the surface $z = f(x, y)$ is given by the formula:*

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

## 4. Center of Gravity

*The coordinates of the center of gravity of a planar figure are given by the formulas:*

$$x_C = \frac{\iint_D x\rho(x, y)\, dx\, dy}{\iint_D \rho(x, y)\, dx\, dy},$$
$$y_C = \frac{\iint_D y\rho(x, y)\, dx\, dy}{\iint_D \rho(x, y)\, dx\, dy}.$$

***Example 1.8*** *Calculate the mass $m$ and the coordinates of the center of gravity $(x_c, y_c)$ of the plate bounded by the lines $y = -x+3$, $y = \frac{x}{2}$, and $x = 0$, with the given density function:*

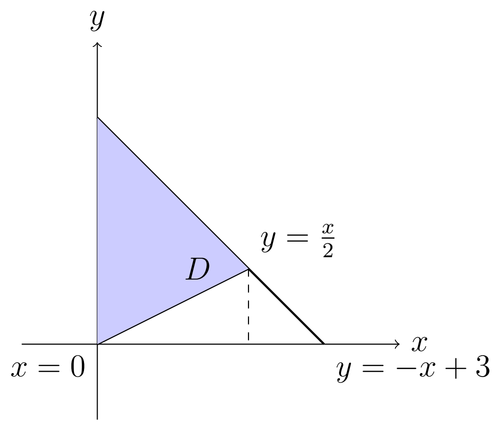
$$\rho(x, y) = 3(x + y).$$

*Consider the domain*

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid y = -x + 3,\ y = \frac{x}{2},\ x = 0\right\}.$$

*The mass $m$ is calculated as follows:*

$$m = \iint_D \rho(x, y)\, dx\, dy,$$




*where* $\rho(x,y) = 3(x+y)$. *Thus,*

$$
\begin{aligned}
m &= \int_0^2 \int_{x/2}^{-x+3} (3x+3y)\,dy\,dx \\
&= \int_0^2 \left[ 3x \left( y \Big|_{x/2}^{-x+3} \right) + 3 \left( \frac{y^2}{2} \Big|_{x/2}^{-x+3} \right) \right] dx \\
&= \int_0^2 \left[ 3x \left( (-x+3) - \frac{x}{2} \right) + 3 \left( \frac{(-x+3)^2 - \left(\frac{x}{2}\right)^2}{2} \right) \right] dx \\
&= \int_0^2 \left[ 3x \left( -x + 3 - \frac{x}{2} \right) + \frac{3}{2} \left( (-x+3)^2 - \frac{x^2}{4} \right) \right] dx \\
&= \int_0^2 \left[ 3x \left( -\frac{3x}{2} + 3 \right) + \frac{3}{2} \left( x^2 - 6x + 9 - \frac{x^2}{4} \right) \right] dx \\
&= \int_0^2 \left[ -\frac{9x^2}{2} + 9x + \frac{3}{2} \left( \frac{3x^2}{4} - 6x + 9 \right) \right] dx \\
&= \int_0^2 \left[ -\frac{9x^2}{2} + 9x + \frac{9x^2}{8} - 9x + \frac{27}{2} \right] dx \\
&= \int_0^2 \left[ -\frac{27x^2}{8} + \frac{27}{2} \right] dx \\
&= \left[ -\frac{27}{8} \cdot \frac{x^3}{3} + \frac{27}{2} x \right]_0^2 \\
&= -\frac{27}{8} \cdot \frac{8}{3} + \frac{27}{2} \cdot 2 \\
&= -\frac{27}{3} + 27 \\
&= 18.
\end{aligned}
$$

*The center of gravity coordinates are given by:*

$$x_C = \iint_D x\rho(x,y)\,dx\,dy \iint_D \rho(x,y)\,dx\,dy.$$

*To find $x_C$:*

$$\begin{aligned}
x_C &= \frac{\int_0^2 \int_{x/2}^{-x+3} x(3x+3y)\,dy\,dx}{18} \\
&= \frac{\int_0^2 \left[ x \left( 3x \left( y\Big|_{x/2}^{-x+3} \right) + 3 \left( \frac{y^2}{2}\Big|_{x/2}^{-x+3} \right) \right) \right] dx}{18} \\
&= \frac{\int_0^2 \left[ x \left( 3x \left( -x+3-\frac{x}{2} \right) + \frac{3}{2} \left( (-x+3)^2 - \frac{x^2}{4} \right) \right) \right] dx}{18}.
\end{aligned}$$

$$\begin{aligned}
&* \iint_0 x\rho(x,y)dxdy \\
&= \int_0 \int_0^2 x(3x+3y)dxdy \\
&= \int_0^1 \int_{x/2}^{-x+3} \left(3x^2+3xy\right) dxdy = \frac{27}{2} = 13.5
\end{aligned}$$

$$\begin{aligned}
&x_c = \frac{13,5}{18} = 0.75 \\
&* y_C = \frac{\iint_0 y\rho(x,y)dxdy}{\iint_D f(x,y)dxdy} \\
&* \iint_0 y\rho(x,y)dxdy = \\
&= \iint_0 y(3x+3y)dxdy \\
&= \int_0^0 \int_{x_2}^{-x+3} 3x+3y+3y^2 dxdy = 27.
\end{aligned}$$

*So:* $y_c = \frac{27}{18} = 1.5.$

$$(x_c, y_c) = (0,75, 1.5)$$

## 1.4 Triple Integrals

***Definition 1.11*** *Let $f : D \to \mathbb{R}$ be a function defined on a domain $D \subset \mathbb{R}^3$. The integral of $f$ over $D$ is called a triple integral and is denoted by:*

$$\iiint_D f = \iiint_D f(x, y, z)\, dx\, dy\, dz.$$

***Remark 1.3*** *Triple integrals share many of the algebraic properties of double integrals. In particular, the computation of triple integrals involves using the Fubini's theorem.*

### 1.4.1 Triple Integral Calculations

***Theorem 1.5 (Fubini's Theorem)*** *states that if $f$ is a continuous function on the rectangular domain $D = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3$, then the triple integral of $f$ over $D$ can be computed as:*

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \int_e^f \left[\int_c^d \left(\int_a^b f(x, y, z)\, dx\right) dy\right] dz.$$

***Example 1.9*** *Calculate the integral:*

$$I = \iiint_\Omega (x + 3yz)\, dx\, dy\, dz,$$

*Consider the integral over the domain $D$:*

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 1 \le y \le 2,\ 1 \le z \le 3\}.$$

*Calculate the integral:*

$$\begin{aligned}
I &= \int_1^3 \int_1^2 \int_0^1 (x + 3yz)\, dx\, dy\, dz \\
&= \int_1^3 \int_1^2 \left[\frac{1}{2} + 3xyz\right]_0^2 dy\, dz \\
&= \int_1^2 \int_1^3 \left[\frac{1}{2}y + \frac{3}{2}zy^2\right]_1^2 dy\, dz \\
&= \int_1^2 \int_1^3 \left(\frac{1}{2} + 6z - \frac{3}{2}z\right) dy\, dz \\
&= \int_1^2 \left[\frac{1}{2}z + 3z^2 - \frac{3}{4}z^2\right]_1^3 dz \\
&= 19.
\end{aligned}$$

***Remark 1.4*** *For a domain $D$ defined as:*

$$D = \left\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Delta,\ u(x,y) \leq z \leq v(x,y)\right\},$$

*where $u$ and $v$ are continuous functions and $\Delta$ is a bounded subset of $\mathbb{R}^2$, the triple integral is given by:*

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_\Delta \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

***Example 1.10*** *Calculate the integral:*

$$I = \iiint_D dx\,dy\,dz$$

*where $D$ is given by:*

$$D = \left\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Delta^2 = [0,1]^2, 0 \leq z \leq x\right\}.$$

*To compute this integral:*

$$\begin{aligned} I &= \iiint_D z\,dx\,dy\,dz \\ &= \iint_\Delta (\int_0^x z)\,dx\,dy\,dz \quad = 1/2. \end{aligned}$$

***Remark 1.5*** *In the case where:*

$$D = \left\{(x,y,z) \in \mathbb{R}^3 \mid x_1 \leq x \leq x_2,\ y_1(x) \leq y \leq y_2(x),\ z_1(x,y) \leq z \leq z_2(x,y)\right\}$$

*the triple integral of the function $f(x,y,z)$ over $D$ is given by:*

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

***Example 1.11*** *Calculate the integral:*

$$I = \iiint_D z\,dx\,dy\,dz$$

*where $D$ is given by:*

$$D = \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq \frac{1}{2},\ x \leq y \leq 2x,\ 0 \leq z \leq \sqrt{1-x^2-y^2}\right\}.$$

## Change of Variables

### A. General Case

*Consider the change of variables given by*

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

*where* $x(u, v, w)$, $y(u, v, w)$, *and* $z(u, v, w)$ *and their partial derivatives are continuous functions, and the Jacobian determinant* $J$ *is non-zero:*

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0.$$

*Then, the triple integral of* $f(x, y, z)$ *over* $D$ *can be expressed as:*

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_\Delta f(x(u, v, w), y(u, v, w), z(u, v, w))\, |J|\, du\, dv\, dw.$$

## B. Cylindrical Coordinates

*In cylindrical coordinates, the transformations are given by:*

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

*In this coordinate system, the Jacobian determinant* $J$ *for the transformation is* $r$, *which accounts for the change in volume element when integrating. Hence, the integral in cylindrical coordinates becomes:*

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_\Delta f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

*The triple integral of a function $f(x, y, z)$ can be calculated using the following formula:*

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_\Delta f(r\cos\theta, r\sin\theta, z)\,|J|\,dr\,d\theta\,dz$$

*where $J$ is the Jacobian determinant of the coordinate transformation.*

***Example 1.12*** *Calculate the integral:*

$$I = \iiint_D z\sqrt{x^2+y^2}\,dx\,dy\,dz$$

*where the domain $D$ is bounded by the cylinder $x^2+y^2 = 2z$, the plane $y = 0$, and the planes $z = 0$ and $z = a$.*

*Using cylindrical coordinates, we have:*

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

*Thus, the integrand $z\sqrt{x^2+y^2}$ simplifies to:*

$$z\sqrt{x^2+y^2} = z\sqrt{r^2} = 3r$$

*The volume element in cylindrical coordinates is:*

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

*n cylindrical coordinates:*

$$x^2 + y^2 = 2z$$
$$r^2 = 2z$$
$$so \quad r = 2z$$
$$Thus, \quad 0 \le r \le 2cos\theta$$
$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le z \le a$$

*We obtain:*

$$\begin{aligned}
&\iiint_D z\sqrt{x^2+y^2}\,dx\,dy\,dz \\
&= \iiint_D zr\,r\,dr\,d\theta\,dz \\
&= \int_0^a \int_0^{2\pi} \int_0^{\sqrt{2z}} zr^2\,dr\,d\theta\,dz \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} r^2 \int_0^a z\,dz\,dr\,d\theta \\
&= \frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= \frac{4}{3}a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta \\
&= \frac{4}{3}a^2 \int_0^{\pi/2} \left(1-\sin^2\theta\right)\cos\theta\,d\theta \\
&= \frac{4}{3}a^2 \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} \\
&= \frac{8}{9}a^2.
\end{aligned}$$

## C. Spherical Coordinates

*The spherical coordinates are given by:*

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\sin\varphi \end{cases}$$

*where*

$$\begin{cases} 0 \le r < \infty \\ 0 \le \theta < 2\pi \\ -\frac{\pi}{2} \le \varphi < \frac{\pi}{2} \end{cases}$$

*The Jacobian of the transformation is* $J = r^2\cos\varphi$.

***Example 1.13*** *Calculate:*

$$I = \iiint_D dx\,dy\,dz$$

$$D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\right\}.$$

*The region $\Delta$ in spherical coordinates is defined as:*

$$\Delta = \left\{ \ (r, \theta, \varphi) \in \mathbb{R}^3 \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -\tfrac{\pi}{2} \leq \varphi \leq \tfrac{\pi}{2} \ \right\}$$

*The integral is:*

$$\begin{aligned} I &= \iiint r^2 |\cos\varphi| \, dr \, d\varphi \, d\theta \\ &= \left( \int_0^1 r^2 \, dr \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi \right) \\ &= \frac{1}{3} \times 2\pi \times 2 = \frac{4}{3}\pi \end{aligned}$$

### 1.4.2 Applications of Integration

## 1. Volume Calculation

*To calculate the volume of a domain $D$, we use:*

$$V = \iiint dx \, dy \, dz$$

## 2. Calculation of Mass and Centroid

*Let $\rho(x, y, z)$ be the density of the material in a certain domain $D$. The mass $M$ is calculated using the formula:*

$$M = \iiint_D \rho(x, y, z) \, dx \, dy \, dz$$

*The coordinates of the centroid are determined using the formulas:*

$$\begin{aligned} \bar{x} &= \frac{1}{M} \iiint_D x\rho(x, y, z) \, dx \, dy \, dz \\ \bar{y} &= \frac{1}{M} \iiint_D y\rho(x, y, z) \, dx \, dy \, dz \\ \bar{z} &= \frac{1}{M} \iiint_D z\rho(x, y, z) \, dx \, dy \, dz \end{aligned}$$

***Example 1.14*** *Calculate the coordinates of the centroid of a prismatic solid bounded by the planes $x = 0$, $z = 0$, $y = 1$, $y = 3$, and $x + 2z = 3$.*

*The volume $V$ is calculated as follows:*

$$V = \iiint_D dx \, dy \, dz$$

*The domain $D$ is bounded by the planes $x = 0$, $z = 0$, $y = 1$, $y = 3$, and $x + 2z = 3$. To set up the integral, solve for the bounds:*

$$V = \int_0^3 \int_1^3 \int_0^{\frac{3-x}{2}} dz\, dy\, dx$$

*Here, $\rho(x, y, z) = 1$ as the density is uniform.*

# Chapter 2

# Improper integrals

## 2.1 Integrals of functions defined over an unbounded interval

*Our goal in this chapter is to compute integrals over unbounded intervals (ranging up to* $-\infty$*, 0, or* $+\infty$*), or integrals on a bounded interval of functions having an infinite limit at some point of the integration interval, a type of integral called an improper integral. Example:* $\int_{-1}^{1} \frac{1}{x} dx =?$

*Why is the answer unclear?*

***Definition 2.1*** *The integral defined by* $\int_a^b f(x)dx$ *is called an improper or generalized integral if:*

1. $f(x)$ *tends toward infinity at one or more points in* $[a, b]$ *(point of infinite discontinuity).*
2. *At least one of the integration bounds is infinite* $(\pm\infty)$*.*

***Example 2.1*** *(a)* $\int_1^{+\infty} \frac{dt}{t^2+1}$ *is improper at* $+\infty$*.*

*(b)* $\int_0^1 \frac{dt}{\sqrt{1-t}}$ *is improper at 1.*

*(c)* $\int_0^2 \frac{1}{x} dx$ *is improper at 0.*

*(d)* $\int_7^{\infty} \frac{1}{x-5} dx$ *has an infinite bound.*

***Definition 2.2 (Singular Point)*** *A point $x_0$ is called a singular point for a function $f$ if it is not bounded at that point, i.e., $\lim_{x\to x_0} f(x) = \pm\infty$.*

***Definition 2.3*** *Two integrals are said to be of the same nature if they are either both convergent or both divergent.*

### 2.1.1 Improper Integrals of Functions Defined on an Unbounded Interval

*Let $f : [a,b] \to \mathbb{R}$ be a Riemann-integrable function on $[a,b]$, where at least one of the bounds of the interval $[a,b]$ is infinite.*

***Definition 2.4*** *Let $f$ be a Riemann-integrable function on $[a,+\infty[$. We say that the improper integral $\int_0^{+\infty} f(t)dt$ converges if the limit $\lim_{x\to+\infty}\int_0^x f(t)dt$ is finite. In this case, we define:*

$$\int_0^{+\infty} f(t)dt = \lim_{x\to+\infty}\int_0^x f(t)dt$$

*In the contrary case, if:*

$$\lim_{x\to+\infty}\int_0^x f(t)dt = \pm\infty$$

*we say that the integral diverges.*

***Remark 2.1*** *For $\int_{-\infty}^b f(t)dt$, the definition can be adapted in an obvious way.*

***Example 2.2***

1. *Let $f(t) = \frac{1}{t}$. It is defined and continuous on $[1,+\infty[$. Indeed:*

$$\lim_{x\to\infty}\int_1^x \frac{1}{t}dt = \lim_{x\to\infty}(\ln x - \ln 1) = +\infty$$

*Thus, the generalized or improper integral $\int_1^{+\infty}\frac{1}{t}dt$ diverges.*

2. *Let $f(t) = \frac{1}{(t-1)^2}$. It is defined and continuous on $[2,+\infty[$.*

$$\lim_{x\to\infty}\int_2^x \frac{1}{(t-1)^2}dt = \lim_{x\to\infty}\left(\frac{-1}{x-1}+1\right) = 1$$

*Thus, the generalized integral $\int_2^{+\infty}\frac{1}{(t-1)^2}dt$ converges.*

3. *The integral* $\int_0^{+\infty} \frac{1}{1+t}dt$ *diverges:*

$$\int_0^n \frac{1}{1+t}dt = \lim_{n\to\infty} \ln(1+n) = +\infty$$

*Let* $f(t) = \frac{1}{(t-1)^2}$. *It is defined and continuous on* $]-\infty, 0]$.

$$\lim_{x\to\infty} \int_x^0 \frac{1}{(t-1)^2}\,dt = \lim_{x\to\infty} \left[\frac{-1}{t-1}\right]_x^0 = \lim_{x\to\infty} \left(1 + \frac{1}{t-1}\right) = 1.$$

*Thus, the integral converges.*

## 2.2 Improper integrals of functions defined on a bounded interval:

*Let* $f$ *be a function defined on* $[a, b]$.

***Definition:*** *Let* $f : [a, b[\to \mathbb{R}$ *be integrable on* $[a, b[$. *We say that the improper integral converges if*

$$\lim_{x\to b} \int_a^x f(t)\,dt$$

*is finite. In this case, we write:*

$$\int_a^b f(t)\,dt = \lim_{x\to b} \int_a^x f(t)\,dt.$$

*In the contrary case, we say that the improper integral* $\int_a^b f(t)\,dt$ *diverges.*

***Remark 2.2*** *For a function* $f$ *defined and integrable on* $]a, b]$, *the definition adapts accordingly.*

***Example 2.3*** 1. *Let* $f(t) = \frac{1}{t}$, *defined and continuous on* $]0, 1]$. *Indeed,*

$$\lim_{x\to 0} \int_x^1 \frac{1}{t}\,dt = \lim_{x\to 0}(-\ln x) = +\infty.$$

*Thus,* $\int_0^1 \frac{1}{t}\,dt$ *diverges.*

2. *Let* $f(t) = \frac{1}{\sqrt{-t}}$, *a function defined on* $[-1, 0[$ *and integrable in the sense of Riemann on* $[-1, 0[$. *Indeed,*

$$\lim_{x\to 0} \int_{-1}^x \frac{1}{\sqrt{-t}}\,dt = \lim_{x\to 0}(-2\sqrt{-x} + 2) = 2.$$

*Thus, the improper integral converges on* $[-1, 0[$.

***Proposition 2.1*** *Let $f$ be a function defined and integrable on $[a, b]$*

*Let $c \in ]a, b[$ be a singular point, i.e., the function $f$ is defined on $[a, b] \setminus \{c\}$. We say that:*

$$\int_a^b f(t)\,dt$$

*converges if both improper integrals,*

$$\lim_{x \to c^-} \int_a^x f(t)\,dt \quad \textit{and} \quad \lim_{x \to c^+} \int_x^b f(t)\,dt,$$

*are finite. In this case, we define:*

$$\int_a^b f(t)\,dt = \lim_{x \to c^-} \int_a^x f(t)\,dt + \lim_{x \to c^+} \int_x^b f(t)\,dt.$$

*Otherwise, we say the integral diverges.*

*[Chasles' Relation]*

*Let $-\infty \leq a < b$ (resp. $a < b \leq +\infty$) and let $f : [a, b[ \to \mathbb{R}$ be continuous on $[a, b[$ (resp. $]a, b]$). Suppose the improper integral $\int_a^b f(t)\,dt$ converges. Let $c \in ]a, b[$. Then:*

1. *$\int_a^c f(t)\,dt$ converges.*
2. *Moreover, we have:*
$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

### 2.2.1 Positivity of the Improper Integral:

***Proposition 2.2*** *Let $f, g : [0, +\infty[ \to \mathbb{R}$ be continuous functions. If both functions have convergent improper integrals and $f \leq g$, then:*

$$\int_a^{+\infty} f(x)\,dx \leq \int_a^{+\infty} g(x)\,dx.$$

*In particular, for a non-negative function $f$, if $f \geq 0$, then:*

$$\int_0^{+\infty} f(x)\,dx \geq 0.$$

***Corollary 2.1 (Symmetry of the Improper Integral)*** *Let $a \in \mathbb{R} \cup \{+\infty\}$, and let $f : [-a, a] \to \mathbb{R}$ be continuous on $] - a, a[$.*

*1.*

*If $f$ is even, then:*

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

*If $f$ is odd, then:*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

### 2.2.2 Convergence Criteria

## 1. Improper Bounded Integral

***Theorem 2.1*** *Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$. Let $f : [a, b[\to \mathbb{R}$ be a piecewise continuous function with $f \geq 0$. The improper integral $\int_a^b f(t)dt$ is convergent if and only if:*

$$\exists M \in \mathbb{R}_+, \quad \forall x \in [a, b[, \quad F(x) = \int_a^x f(t)dt \leqslant M.$$

***Example 2.4*** *Let's study the behavior of the integral $I$ with:*

$$I = \int_0^2 \cos^2\left(\frac{1}{t}\right) dt.$$

*Thus, the integral $\int_0^2 \cos^2\left(\frac{1}{t}\right) dt$ converges.*

## 2. Riemann Integrals

*A Riemann integral is:*

$$\int_1^{+\infty} \frac{1}{t^2} dt.$$

*In this case, the antiderivative is explicit:*

$$\int_1^{+\infty} \frac{1}{t^\alpha} dt = \begin{cases} \lim_{x\to\infty} \left[\frac{1}{-\alpha+1} t^{-\alpha+1}\right]_1^x, & \text{if } \alpha \neq 1, \\ \lim_{x\to+\infty} [\ln t]_1^x, & \text{if } \alpha = 1. \end{cases}$$

*We can immediately deduce the nature of Riemann integrals: if $\alpha > 1$, then $\int_1^{+\infty} \frac{1}{t^\alpha} dt$ converges; if $\alpha \leqslant 1$, then $\int_1^{+\infty} \frac{1}{t^\alpha} dt$ diverges.*

***Theorem 2.2 (Comparison Criterion)*** *Let $a < b \leqslant +\infty$ (or $-\infty < a < b$). Let $f$ and $g : [a, b[\to \mathbb{R}$ be two continuous functions on $[a, b[$ (or $]a, b]$). Suppose:*

$$\forall x \in [a, b[, \quad 0 \leqslant f(x) \leqslant g(x).$$

*Then:*

- *If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.*
- *If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.*

*Furthermore, in the case where both converge, we have:*

$$0 \leqslant \int_a^b f(x)dx \leqslant \int_a^b g(x)dx.$$

***Remark 2.3*** *The comparison criterion is not applicable to functions that change sign.*

***Example 2.5*** *We aim to determine the convergence of:*

$$I = \int_0^{+\infty} \frac{\sin^2(t)}{1+t^2} dt.$$

*The function $f(t) = \frac{\sin^2(t)}{1+t^2}$ is continuous, hence integrable in the Riemann sense over $[0, +\infty[$.*

*We have a potential issue with convergence at $t \to +\infty$. However, since the function is positive, we can apply the comparison criterion.*

*We know that:*

$$0 \leq \frac{\sin^2(t)}{1+t^2} \leq \frac{1}{1+t^2} \quad \forall t \in [0, +\infty[.$$

*Since the integral $\int_0^{+\infty} \frac{1}{1+t^2} dt$ converges, due to the fact that its antiderivative is* $\arctan(t)$ *and* $\arctan(t) \to \frac{\pi}{2}$ *as $t \to +\infty$, we conclude that:*

$$\int_0^{+\infty} \frac{\sin^2(t)}{1+t^2} dt$$

*also converges by the comparison test.*

## 3. Absolutely Convergent Integral

*Let* $f : [a, b[\to \mathbb{R}$ *be a function that is integrable on* $[a, b[$. *We say that the improper integral* $\int_a^b f(t)dt$ *converges absolutely if and only if:*

$$\int_a^b |f(t)|dt \quad \textit{converges.}$$

***Proposition 2.3*** *If* $\int_a^b |f(t)|dt$ *converges, then* $\int_a^b f(t)dt$ *also converges and we have the inequality:*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

***Example 2.6*** *Consider the integral:*

$$I = \int_1^{+\infty} \frac{\sin(t)}{t^2}dt.$$

*We define the function* $f(t) = \frac{\sin(t)}{t^2}$. *This function is continuous on* $[1, +\infty[$, *and for all* $t \in [1, +\infty[$, *we have:*

$$|f(t)| = \left|\frac{\sin(t)}{t^2}\right| \leq \frac{1}{t^2}.$$

*Since the integral* $\int_1^{+\infty} \frac{1}{t^2}dt$ *converges (this is a standard Riemann integral), we conclude that:*

$$\int_1^{+\infty} \frac{\sin(t)}{t^2}dt$$

*also converges.*