1^{st} Tutorial Series
"Simple and Multiple Integrals"

2^{nd} Year Engineering (S3)

Exercise 1:

1. Using the definition of the Riemann integral, calculate:

$$I_1=\int_0^1 x dx, I_2=\int_0^1 x^2 dx, I_3=\int_0^1 x^3 dx, I_4=\int_0^2 (3x-1) dx, \quad I_5=\int_{-1}^3 \left(x^2+6x+3\right) dx$$

(Hint: $\sum_{k=1}^n k=\frac{n(n+1)}{2}, \sum_{k=1}^n k^2=\frac{n(n+1)(2n+1)}{6}, \sum_{k=1}^n k^3=\frac{1}{4}n^2(n+1)^2.$)

2. Verify the results by calculating the integrals in a straightforward manner.

Exercise 2:

Using the Riemann sum formula, calculate the limits:

$$\lim_{n\to+\infty}\sum_{k=1}^n \frac{1}{2n+k}, \quad \lim_{n\to+\infty}\sum_{k=1}^n \frac{1}{3n+k}, \quad \lim_{n\to+\infty}\sum_{k=1}^n \frac{1}{\sqrt{n^2+2nk}},$$
$$\lim_{n\to+\infty}\sum_{k=1}^n \frac{k}{n^2+k^2}, \quad \lim_{n\to+\infty}\sum_{k=1}^n \frac{n}{n^2+k^2}.$$

Exercise 3:

Calculate the following integrals:

- $\int_0^{\frac{\pi}{2}} x\sin(x)\, dx$. (By parts)
- $\int_0^1 x\sqrt{1-x^2}dx$, $\quad \int_{\sqrt{\pi}}^{2\sqrt{\pi}} 2x\cos\left(x^2\right) dx$, $\quad \int \cos^4(x)\sin^3(x)\, dx$. (Change of variable)
- $\int_0^1 \frac{x-1}{x^2-4}dx$.(By decomposition)

Exercise 4:

Calculate the following integrals:

1. $I_1=\iint_{D_1} x dxdy, \quad D_1=\left\{(x,y)\in \mathbf{R}^2, 4-x^2\geq y\geq 0\right\}$,
2. $I_2=\iint_{D_2} e^x dxdy, \quad D_2=\left\{(x,y)\in \mathbf{R}^2, y\geq x^2, 2x\geq y\right\}$,
3. $I_3=\iint_{D_3} y dxdy, \quad D_3=\left\{(x,y)\in \mathbf{R}^2, x^2+y^2\leq 4, x\geq 0, y\geq 0\right\}$,
4. $I_4=\iint_{D_4} x\ln y dxdy, \quad D_4=\left\{(x,y)\in \mathbf{R}^2, x>-2, y>1, x<3, y<4\right\}$,

Exercise 5:

Calculate the following integrals using Fubini's theorem:
$I_1 = \int_0^1 \int_0^1 (2x+y)dxdy, \quad I_2 = \int_1^2 \int_3^5 xydxdy, \quad I_3 = \int_0^1 \int_0^1 \frac{2x+y}{1+x^2}dxdy$

Exercise 6:

Calculate the following integrals using a change of variables:

1. $I_1 = \iint_{D_1} xydxdy, \quad D_1 = \{(x,y) \in \mathbf{R}^2, x^2+y^2 \leq 1, x>0, y>0\}$,
2. $I_2 = \iint_{D_2} (x-1)^2 dxdy, \quad D_2 = \{(x,y) \in \mathbf{R}^2, -1 \leq x+y \leq 1, -2 \leq x-y \leq 2\}$,
3. $I_3 = \iint_{D_3} ydxdy, \quad D_3 = \{(x,y) \in \mathbf{R}^2, x^2+y^2 \leq 4, x \geq 0, y \geq 0\}$,

Exercise 7:

Calculate the area of the following regions:

1. $D_1 = \{(x,y) \in \mathbf{R}^2, y = 5 - x^2, y \geq 1\}$
2. $D_2 = \{(x,y) \in \mathbf{R}^2, x^2+y^2 \leq 1, x>0, y>0\}$,
3. $D_3 = \{(x,y) \in \mathbf{R}^2, x \geq 0, y = \sin x, y = \cos x\}$
4. $D_1 = \{(x,y) \in \mathbf{R}^2, y \leq 2 - x^2, y \geq x-1\}$

Exercise 8:

Calculate the following integrals:

1. $I_1 = \iiint_{D_1} x^2 y e^{xyz} dxdydz, \quad D_1 = [0,1] \times [0,2] \times [-1,1]$,
2. $I_2 = \iiint_{D_2} x^2 dxdydz, \quad D_2 = \{(x,y,z) \in \mathbf{R}^3, x^2+y^2+z^2 \leq R^2\}$,
3. $I_3 = \iiint_{D_3} \sqrt{x^2+y^2+z^2} dxdydz, \quad D_3 = \{(x,y,z) \in \mathbf{R}^3, x^2+y^2+z^2 \leq 1\}$,
4. $I_3 = \iiint_{D_4} \frac{1}{(x+y+z+1)^3} dxdydz, \quad D_3 = \{(x,y,z) \in \mathbf{R}^3, x+y+z=1, x=0, y=0, z=0\}$.

Exercise 9:

Calculate the volume of the following regions:

1. $D_1 = \{(x,y,z) \in \mathbf{R}^3, x^2+y^2+z^2 \leq 1\}$
2. $D_2 = \left\{(x,y,z) \in \mathbf{R}^3, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$.