Chapter 1

Logic Notions

1.1 Notions

Definition 1.1.1

- We call any relation P that is either true or false a "logical proposition".
- When the proposition is true, it is assigned the value 1.
- When the proposition is false, it is assigned the value 0.
- These values are called "Truth values of the proposition".

Example 1.1.2

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1. "I am taller than you", is a proposition.
2. "$2 + 2 = 4$" is a proposition.
3. "$2 \times 3 = 7$" is a proposition.
4. "For all $x \in \mathbb{R}$; we have $x^2 \geq 0$" is a proposition.
5. "How are you today ?" is not a proposition.

Thus, to define a logical proposition, it suffices to give its truth values. Generally, these values are put into a table called a "Truth table".

1.1.1 Logical Operations

Definition 1.1.3 Negation : "$\overline{P}$"

Given a logical proposition P, we call the negation of P the logical proposition $\overline{P}$, which is false when P is true and true when P is false, so we can represent it as follows :

P	$\overline{P}$
1	0
0	1

Definition 1.1.4 Conjunction : "$\wedge$"

Let P and Q be two logical propositions, we call "conjunction" of P and Q the proposition "$P \wedge Q$ ", which is true when both P and Q are true and false otherwise. Its truth table :

P	Q	$P \wedge Q$
1	1	1
0	0	0
1	0	0
0	1	0

Definition 1.1.5 Disjunction "$\vee$"

Let P and Q be two logical propositions, we call "disjunction" of P and Q the proposition "$P \vee Q$ ", which is true if either of the logical propositions P or Q is true. Its truth table :

P	Q	$P \vee Q$
1	1	1
0	0	0
1	0	1
0	1	1

Definition 1.1.6 Implication "$\Rightarrow$"

Consider two logical propositions P and Q, we denote "$P \Rightarrow Q$" the logical proposition that is false if P is true and Q is false. The proposition P $\Rightarrow$ Q reads "P implies Q ".

P	Q	$P \Rightarrow Q$
1	1	1
0	0	1
1	0	0
0	1	1

Given two logical propositions P and Q, the truth table of $\overline{P} \vee Q$ is as follows :

P	$\overline{P}$	Q	$\overline{P} \vee Q$
1	0	1	1
0	1	0	1
1	0	0	0
0	1	1	1

We see that this table is identical to that of $P \Rightarrow Q$, so we say that the proposition $P \Rightarrow Q$ is equivalent to the proposition $\overline{P} \vee Q$:

Definition 1.1.7 Equivalence "$\Leftrightarrow$"

We say that the two logical propositions P and Q are logically equivalent if they are true simultaneously or false simultaneously, and we denote "$P \Leftrightarrow Q$ ", its truth table is :

P	Q	$P \Leftrightarrow Q$
1	1	1
0	0	1
1	0	0
0	1	0

1.1.2 DeMorgan's Rules

Proposition 1.1.8

Let P and Q be two logical propositions, then :

1. $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$.
2. $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$.

Proof

We establish the proof of these rules by giving the truth values of the corresponding logical propositions.

1.

P	Q	$\overline{P}$	$\overline{Q}$	$P \wedge Q$	$\overline{P \wedge Q}$	$P \vee Q$	$\overline{P \vee Q}$	$\overline{P} \vee \overline{Q}$	$\overline{P} \wedge \overline{Q}$
1	1	0	0	1	0	1	0	0	0
0	0	1	1	0	1	0	1	1	1
1	1	0	1	1	0	1	0	1	0
1	0	0	1	0	1	1	0	1	0
0	1	1	0	0	1	1	0	1	0
1	0	0	0	0	1	1	0	0	0
0	1	1	0	0	1	1	0	1	0
0	0	1	1	0	1	0	1	1	1

2.

P	Q	$\overline{P}$	$\overline{Q}$	$P \vee Q$	$\overline{P \vee Q}$	$\overline{P} \wedge \overline{Q}$
1	1	0	0	1	0	0
0	0	1	1	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0

1.2 Quantifiers

1.2.1 Universal Quantifier $\forall$, or "for all"

A proposition P may depend on a parameter x. For example : "$x^2 \geq 1$ " the assertion $P(x)$ is true or false depending on the value of x.

Definition 1.2.1

$\forall x \in E, P(x)$, is true when the propositions $P(x)$ are true for all elements x in the set E. We read "For all x belonging to E, $P(x)$".

Example 1.2.2

1. $\forall x \in [1; +\infty[\,;\, x^2 \geq 1$ is a true proposition.
2. $\forall x \in \mathbb{R}\,;\, x^2 \geq 1$ is a false proposition.
3. $\forall n \in \mathbb{N}\,;\, n(n+1)$ is divisible by 2 is true.

1.2.2 Existential Quantifier $\exists$, or "there exists"

Definition 1.2.3

$\exists x \in E; P(x)$, is true when we can find at least one x from E for which $P(x)$ is true. We read "there exists x belonging to E such that $P(x)$ is true".

Example 1.2.4

1. $\exists x \in \mathbb{R}\,;\, x(x-1) < 0$ is true.
2. $\exists n \in \mathbb{N}\,; n^2 - n > n$ is true.
3. $\exists x \in \mathbb{R}\,;\, x^2 = -4$ is false.

1.2.3 Negation of Quantifiers

Definition 1.2.5

1. The negation of "$\forall x \in E; P(x)$" is "$\exists x \in E; \overline{P(x)}$".
   For example, the negation of "$\forall x \in \mathbb{R}; x^2 \geq 1$ " is the assertion "$\exists x \in \mathbb{R}; x^2 < 1$".
2. The negation of "$\exists x \in E; P(x)$" is "$\forall x \in E; \overline{P(x)}$".
   For example, the negation of "$\exists n \in \mathbb{N};\, n^2 - n > n$" is the assertion "$\forall n \in \mathbb{N};\, n^2 - n \leq n$".
3. Negation of complex sentences : for example, the proposition "$\forall x \in E,\, \exists y \in E\,;\, P(x;y)$" its negation is "$\exists x \in E,\, \forall y \in E\,;\, \overline{P(x;y)}$".
   For example, the negation of "$\forall x \in \mathbb{R},\, \exists y \in \mathbb{R},\, x + y > 0$" is "$\exists x \in \mathbb{R},\, \forall y \in \mathbb{R},\, x + y \leq 0$".

Note 1.2.6

The order of quantifiers is very important. For example, the two logical sentences

$$\forall x \ \in \mathbb{R}, \exists y \ \in \mathbb{R} x + y > 0,$$

and

$$\exists\, x \ \in \mathbb{R}, \forall\, y \ \in \mathbb{R} x + y > 0,$$

are different.
The first one is true, the second one is false. Indeed, the first sentence asserts that "For every real number x, there exists a real number y (which may depend on x) such that $x + y > 0$" (for example, for a given x we can take $y = -x + 1$). So, it is a true sentence. On the other hand, the second one reads : "There exists a real number y, such that for every real number x, $x + y > 0$." This sentence is false, it cannot be the same y that works for all x.

1.3 Types of Reasoning

Definition 1.3.1 Direct Reasoning

We want to show that the proposition "$P \Rightarrow Q$" is true. We assume that P is true and then show that Q is true.

Example 1.3.2

Show that

$$\forall x, y \in \mathbb{R}^+, \quad x \leq y \Rightarrow x \leq \frac{x+y}{2} \leq y.$$

Proof

We have

$$x \leq y \Rightarrow x + x \leq x + y \Rightarrow 2x \leq x + y \Rightarrow x \leq \frac{x+y}{2}. \tag{1.1}$$

$$y \geq x \Rightarrow y + y \leq x + y \Rightarrow 2y \geq x + y \Rightarrow y \geq \frac{x+y}{2}. \tag{1.2}$$

From (1.1) and (1.2) we have :

$$x \leq \frac{x+y}{2} \leq y.$$

So

$$\forall x, y \in \mathbb{R}^+, \quad x \leq y \Rightarrow x \leq \frac{x+y}{2} \leq y,$$

is true.

Definition 1.3.3 Case by Case

If we want to verify a proposition $P(x)$ for all x in a set E, we show the proposition $P(x)$ for $x \in A \subset E$, and then for $x \notin A$.

Example 1.3.4

Show that : $\forall x \in \mathbb{R}\,;\, |x-1| \leq x^2 - x + 1$.

Proof

1. If

$$x \geq 1, \quad |x-1| = x - 1.$$

Then,

$$x^2 - x + 1 - |x-1| = x^2 - x + 1 - x + 1 = x^2 - 2x + 2 = (x-1)^2 + 1 \geq 0.$$

So

$$x^2 - x + 1 \geq |x-1|\,.$$

2. If

$$x < 1, |x-1| = -(x-1).$$

Then,

$$x^2 - x + 1 - |x-1| = x^2 - x + 1 - (-x+1) = x^2 - x + 1 + x - 1 = x^2 \geq 0.$$

So

$$x^2 - x + 1 \geq |x-1|\,.$$

Conclusion, in all cases

$$\forall x, \quad \in |x-1| \leq x^2 - x + 1.$$

Definition 1.3.5 Contrapositive

Reasoning by "contraposition" is based on the following equivalence :

$$(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P}).$$

So, if we want to show the assertion "$P \Rightarrow Q$" we actually show that if Q is true. Then, P is true.

Example 1.3.6

Show that : $\forall n \in \mathbb{N}\,;\, n^2$ is even then n is even.

Proof

We want to show that if n^2 is odd $\Rightarrow n$ is odd.
$\forall\, n$ is odd, then there exists $k \in \mathbb{N}$ such that $n = 2k+1$ Then, $n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2k^{'}+1$.
So, n^2 is odd.

Definition 1.3.7 Absurd

Reasoning by "absurd" to show that "$P \Rightarrow Q$" is based on the following principle : we assume both that P is true and Q is false. We seek a contradiction. Thus if P is true then Q must be true and therefore "$P \Rightarrow Q$" is true.

Example 1.3.8

Show that : $\forall x, y \in \mathbb{R}^+$. If,

$$\frac{x}{1+y} = \frac{y}{1+x}.$$

Then, $x = y$.

Proof

We assume that $\frac{x}{1+y} = \frac{y}{1+x}$ and $x \neq y$.

Since

$$\frac{x}{1+y} = \frac{y}{1+x}.$$

Then,

$$x(1+x) = y(1+y).$$

So

$$x + x^2 = y + y^2.$$

Hence

$$x^2 - y^2 = -x + y.$$

So,

$$(x-y)(x+y) = -(x-y).$$

Since $x \neq y$. Then, $x - y \neq 0$ and so by dividing by $x - y$ we get $x + y = -1$ this is a contradiction(the sum of two positive numbers is positive). Conclusion, $\forall x, y \in \mathbb{R}^+$. If,

$$\frac{x}{1+y} = \frac{y}{1+x}.$$

Then, $x = y$.

Definition 1.3.9 Counterexample

By counterexample to show that "$\forall x \in E\,;\, P(x)$" is false. It suffices to find $x \in E$, such that $P(x)$ is false.

Example 1.3.10

Show that "every positive integer is the sum of three squares" is false.

Proof

Let's take a counterexample. Consider the integer $n = 7$, the squares less than 7 are 0; 1; 4 but $0+1+4 \neq 7$.

Definition 1.3.11 Recurrence

The principle of "recurrence" allows us to show that a proposition $P(n)$ depending on n, is true for all $n \in \mathbb{N}$. The proof by recurrence proceeds in three steps :

1. **Initialization** : we verify that $P(0)$ is true.
2. **Heredity** : we assume $n > 0$ given with $P(n)$ true. Then, we demonstrate that the proposition $P(n+1)$ at the next rank is true.

3. **Conclusion** : we recall that by the principle of recurrence $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.3.12

Show that

$$\forall n \in \mathbb{N}, \quad P(n) = \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof

1. Initialization : For $n = 0$, we have $0^2 = 0$. So, $P(0)$ is true.
2. Heredity : For $n > 0$, we assume that $P(n)$ is true, i.e

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

is true, and we show that

$$P(n+1) = \sum_{k=0}^{n+1} k^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \sum_{k=0}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6},$$

is true.
$P(n)$ is true so

$$\sum_{k=0}^{n} k^2 = 0^2 + 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We have :

$$\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= 0^2 + 1^2 + 2^2 + ... + n^2 + (n+1)^2, \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2, \\
&= \frac{(n+1)(n+2) + (2n+3)}{6}, \\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

Hence $P(n+1)$ is true.
3. Conclusion,

$$\forall\, n \in \mathbb{N}, \quad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$