SOLUTIONS TO EXERCISES SERIES NO 3

Exercise 1

Reminder 1

1.1 Taylor's formula with Lagrange remainder

For every two numbers $x, x_0$of the interval $[a,b]$ where $x \neq x_0$ we have:

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Where c is a real number from $]x_0, x[$ (or $[x, x_0], if x < x_0$).

a) $x_0 = 1$; $f(x) = \frac{1}{x}$.

$$f^0(x) = f(x) = \frac{1}{x} \;, \quad f'(x) = -\frac{1}{x^2} \;, \quad f''(x) = (-1)^2 \frac{1 \times 2}{x^3} \;, \quad f^{(3)}(x) = (-1)^3 \frac{1 \times 2 \times 3}{x^4}.$$

Using proof by induction we prove that

$$\forall n \in \mathbb{N}: \; f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \quad \text{and} \quad \forall n \in \mathbb{N}: \; f^{(n)}(1) = (-1)^n n!.$$

For all x in $\mathbb{R}_+^*$ where $x \neq 1$ then

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(1)}{j!}(x - 1)^j + \frac{f^{(n+1)}(c)}{(n+1)!}(x - 1)^{n+1},$$

$$= \sum_{j=0}^{n} (-1)^j (x - 1)^j + \frac{(-1)^{n+1}}{c^{n+2}}(x - 1)^{n+1},$$

where c is a real number between x and 1.

b) $x_0 = 0$; $f(x) = xe^{2x}$.

Using the Leibniz formula $\forall n \in \mathbb{N}: (uv)^{(n)} = \sum_{p=0}^{n} C_n^p \, u^{(n-p)} v^{(p)}$ we get

$$\forall n \in \mathbb{N}^*: f^{(n)}(x) = \sum_{p=0}^{n} C_n^p \, (x)^{(p)} (e^{2x})^{(n-p)} \quad \text{and} \quad f^{(0)}(x) = xe^{2x}$$

So

$$\forall n \in \mathbb{N}^*: f^{(n)}(x) = C_n^0 (x)^{(0)} (e^{2x})^{(n)} + C_n^1 (x)^{(1)} (e^{2x})^{(n-1)} + \underbrace{\sum_{p=2}^{n} C_n^p \, (x)^{(p)} (e^{2x})^{(n-p)}}_{=0}$$

$$= (x + n2^{n-1})e^{2x}.$$

So

$$\forall n \in \mathbb{N}^*: f^{(n)}(0) = n2^{n-1} \quad ; \quad f(0) = 0$$

For all x in $\mathbb{R}$ where $x \neq 0$ then

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

$$= \sum_{j=1}^{n} \frac{2^{j-1}}{(j-1)!} x^j + \frac{(c + (n+1)2^n)e^{2c}}{(n+1)!} x^{n+1},$$

where c is a real number between x and 0.

c) $x_0 = 2$; $f(x) = \frac{x^2-2x-1}{x^2-4x+3}$. We have $\frac{x^2-2x-1}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3} + 1$.

Using the proof by induction we prove that

$$\forall n \in \mathbb{N}^*: f^{(n)}(x) = (-1)^n n! \left(\frac{1}{(x-1)^{n+1}} + \frac{1}{(x-3)^{n+1}}\right) \text{ and } f^{(0)}(x) = \frac{1}{x-1} + \frac{1}{x-3} + 1.$$

So

$$\forall n \in \mathbb{N}^*: f^{(n)}(2) = n!\,((-1)^n - 1) \text{ and } f(2) = 1.$$

For all x in $]1,3[$ where $x \neq 2$ then

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(2)}{j!} x^j + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

$$= 1 + \sum_{j=1}^{n} ((-1)^n - 1)\,(x-2)^j + (-1)^{n+1} \left(\frac{1}{(c-1)^{n+2}} + \frac{1}{(c-3)^{n+2}}\right)(x-2)^{n+1}.$$

Since $\sum_{j=1}^{n} ((-1)^n - 1) = \begin{cases} -2\,; & n = 2p+1 \\ 0\,; & n = 2p \end{cases}$ we get

$$f(x) = 1 - 2(x-2) - 2(x-2)^3 - 2(x-2)^5 + \cdots + \left(\frac{1}{(c-1)^{n+2}} + \frac{1}{(c-3)^{n+2}}\right)(2-x)^{n+1},$$

where c is a real number between x and 2.

Exercise 2

a) $\forall x \in \mathbb{R}_+: x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$.

Using Taylor-Lagrange formula for $f(x) = \ln(1+x)$, $x_0 = 0$ and $n = 2$; $n = 3$ we get

For $n = 2 \Longrightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{1}{3}\frac{1}{(c+1)^3}x^3$ where $x \in \mathbb{R}_+$ and c is a real number between x and 0.

For $n = 3 \Longrightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{4}\frac{1}{(k+1)^3}x^4$ where $x \in \mathbb{R}_+$ and k is a real number between x and 0.

Since $\forall x, c, k \in \mathbb{R}_+ : -\frac{1}{4}\frac{1}{(k+1)^3}x^4 \leq 0$ and $\frac{1}{3}\frac{1}{(c+1)^3}x^3 \geq 0$. we get

$$\forall x \in \mathbb{R}_+ : x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.$$

b) $\forall x \in \left[0, \frac{\pi}{2}\right] : x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$

Using Taylor-Lagrange formula for $f(x) = \sin x$, $x_0 = 0$ and $n = 3$; $n = 5$ we get

For $n = 3 \Longrightarrow \sin x = x - \frac{x^3}{6} + \frac{\sin c}{120}x^4$ where $x \in \left[0, \frac{\pi}{2}\right]$ and c is a real number between x and 0.

For $n = 5 \Longrightarrow \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{\sin k}{120}x^6$ where $x \in \left[0, \frac{\pi}{2}\right]$ and k is a real number between x and 0.

Since $\forall x, c, k \in \left[0, \frac{\pi}{2}\right] : -\frac{\sin k}{120}x^6 \leq 0$ and $\frac{\sin c}{120}x^4 \geq 0$. we get

$$\forall x \in \left[0, \frac{\pi}{2}\right] : x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$$

2) Determine the local Extrema of f in each of the following cases.

Reminder 2

2.1 Apply Tyler's formulas to find local extrema

If f is a function of class C^n in the neighborhood of the point x_0 such that: $f'(x_0) = f''(x_0) = f^{(3)}(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Then f accepts a local maximum (local minimum, respectively) at x_0 if and only if n is even and $f^{(n)}(x_0) < 0$ ($f^{(n)}(x_0) > 0$, respectively).

a) $f(x) = x^3 - 2ax^2 + a^2x$ $(a > 0)$.

Necessary condition:

$$f'(x) = 0 \Leftrightarrow 3x^2 - 4ax + a^2 = 0 \Leftrightarrow x = a \; ; \; x = \frac{a}{3}$$

Sufficient condition:

$$f''(a) = 2a > 0 \Rightarrow f(a) = 0 \text{ is local minimum value of } f.$$

$$f''\left(\frac{a}{3}\right) = -2a < 0 \Rightarrow f(a) = \frac{4}{27}a^3 \text{ is local maximum value of } f.$$

Exercise 3

Reminder 3

3.1 Limited Development of order n in a neighborhood of 0

We say that f admits a limited Development of order n in a neighborhood of 0 if and only if there exists a neighborhood v of 0 and constant numbers $a_0, a_1, a_2, \dots\dots\dots, a_n$ where

$$\forall x \in v\,;\, x \neq 0: f(x) = a_0 + a_1 x + a_2 x^2 \dots\dots\dots + a_n x^n + x^n \varepsilon(x) \text{ with } \lim_{x\to 0} \varepsilon(x) = 0.$$

3.2 Operation on Limited Development

Let f, g be two functions admitting limited developments to the same order n in the neighborhood of 0. We denote their regulars parts as $P_n(x)$, $Q_n(x)$, respectively. That is

$$f(x) = P_n(x) + x^n \varepsilon_1(x) \quad ; \quad g(x) = Q_n(x) + x^n \varepsilon_2(x).$$

Then, the functions $f+g, fg, \frac{f}{g}$ (if $\lim_{x\to 0} g(x) \neq 0$), fog (if $\lim_{x\to 0} g(x) = 0$), admitting limited developments of order n in the neighborhood of 0 and we have:

1) $f(x) + g(x) = P_n(x) + Q_n(x) + x^n \varepsilon_3(x)$ with $\lim_{x\to 0} \varepsilon_3(x) = 0$.

2) $f(x)g(x) = A_n(x) + x^n \varepsilon_4(x)$ with $\lim_{x\to 0} \varepsilon_4(x) = 0$.

Where $A_n(x)$ is the polynomial we obtain by retaining in the multiplication $P_n(x)Q_n(x)$ only the terms with degrees less than or equal to n.

3) $\frac{f(x)}{g(x)} = B_n(x) + x^n \varepsilon_5(x)$ with $\lim_{x\to 0} \varepsilon_5(x) = 0$.

Where $B_n(x)$ is the polynomial we obtain by Euclidean division of $P_n(x)$ by $Q_n(x)$ according to increasing powers of x keeping only terms with degrees less than or equal to n.

4) $fog(x) = C(x) + x^n \varepsilon_6(x)$ with $\lim_{x\to 0} \varepsilon_6(x) = 0$.

Where $C_n(x)$ is the polynomial we obtain by retaining in the composite $P_n o\, Q_n(x)$ only the terms with degrees less than or equal to n.

Find a limited Development of order n in a neighborhood of 0 for the functions f in each of the following cases

a) $n = 6$, $f(x) = \frac{x^2+x-1}{x^2+2}$.

We have $x^2 + x - 1 = x^2 + x - 1 + o(x^6)$ and $x^2 + 2 = x^2 + 2 + o(x^6)$ because $x^2 + x - 1$ and $x^2 + 2$ are polynomials.

By Euclidean dividing of $-1 + x + x^2$ over $2 + x^2$, according to increasing powers of x, keeping only terms with degrees less than or equal to 6, we get

$$
\begin{array}{r|l}
-1+x+x^2 & 2+x^2 \\ \cline{2-2}
-1-\frac{1}{2}x^2 & -\frac{1}{2}+\frac{1}{2}x+\frac{3}{4}x^2-\frac{1}{4}x^3-\frac{3}{8}x^4+\frac{1}{8}x^5+\frac{3}{16}x^6 \\ \cline{1-1}
x+\frac{3}{2}x^2 & \\
x+\frac{1}{2}x^3 & \\ \cline{1-1}
\frac{3}{2}x^2-\frac{1}{2}x^3 & \\
\frac{3}{2}x^2+\frac{3}{4}x^4 & \\ \cline{1-1}
-\frac{1}{2}x^3-\frac{3}{4}x^4 & \\
-\frac{1}{2}x^3-\frac{1}{4}x^5 & \\ \cline{1-1}
-\frac{3}{4}x^4+\frac{1}{4}x^5 & \\
-\frac{3}{4}x^4-\frac{3}{8}x^6 & \\ \cline{1-1}
\frac{1}{4}x^5+\frac{3}{8}x^6 & \\
\frac{1}{4}x^5+0 & \\ \cline{1-1}
\frac{3}{8}x^6 & \\
\frac{3}{8}x^6 & \\ \cline{1-1}
0 &
\end{array}
$$

So

$$\frac{-1+x+x^2}{2+x^2} = -\frac{1}{2}+\frac{1}{2}x+\frac{3}{4}x^2-\frac{1}{4}x^3-\frac{3}{8}x^4+\frac{1}{8}x^5+\frac{3}{16}x^6+o(x^6).$$

b) $n = 3\,, f(x) = e^x\sqrt{1-x}$.

We have

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3) \quad \text{and} \quad \sqrt{1-x} = 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+o(x^3).$$

So

$$e^x\sqrt{1-x} = \left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3\right)\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\right),$$

$$= 1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{13}{48}x^3+o(x^3).$$

d) $n = 4\,, f(x) = \ln\left(x+\sqrt{\cos x}\right)$.

We have

$$\sqrt{1+h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3 - \frac{5}{128}h^4 + o(h^4) \quad \text{and} \quad \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4).$$

So

$$x + \sqrt{\cos x} = x + \sqrt{1 + (\cos x - 1)}$$

$$= x + \left[1 + \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right) - \frac{1}{8}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right)^2 + 0 + 0\right]$$

$$= x + \left[1 + \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right) - \frac{1}{8}\left(-\frac{1}{2}x^2\right)^2 + 0 + 0\right]$$

$$= 1 + x - \frac{1}{4}x^2 - \frac{1}{96}x^4 + o(x^4).$$

We have also

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + o(t^4).$$

So

$$\ln\left(x + \sqrt{\cos x}\right) = \ln\left(1 + \left(x + \sqrt{\cos x} - 1\right)\right)$$

$$= \left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right) - \frac{1}{2}\left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right)^2 + \frac{1}{3}\left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right)^3 - \frac{1}{4}\left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right)^4$$

$$= \left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right) - \frac{1}{2}\left(x - \frac{1}{4}x^2\right)^2 + \frac{1}{3}\left(x - \frac{1}{4}x^2\right)^3 - \frac{1}{4}(x)^4$$

$$= \left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4\right) - \frac{1}{2}\left(x^2 - 2x\left(-\frac{1}{4}x^2\right) + \left(-\frac{1}{4}x^2\right)^2\right) + \frac{1}{3}\left(x^3 + 3x^2\left(-\frac{1}{4}x^2\right)\right) - \frac{1}{4}(x)^4.$$

So

$$\ln\left(x + \sqrt{\cos x}\right) = x - \frac{3}{4}x^2 + \frac{7}{12}x^3 - \frac{13}{24}x^4 + o(x^4).$$

e) $n = 3$, $f(x) = (\cos x)^{\frac{1}{x}}$. we have $(\cos x)^{\frac{1}{x}} = e^{\frac{\ln\cos x}{x}}$ so

$$\frac{\ln\cos x}{x} = \frac{1}{x}\ln\left(1 + (\cos x - 1)\right)$$

$$= \frac{1}{x}\left[\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right) - \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right)^2 + 0\right]$$

$$= \frac{1}{x}\left[\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right) - \frac{1}{2}\left(-\frac{1}{2}x^2\right)^2 + 0\right]$$

$$= -\frac{1}{2}x - \frac{1}{12}x^3 + o(x^3).$$

So

$$(\cos x)^{\frac{1}{x}} = e^{\frac{\ln\cos x}{x}} = 1 + \left(-\frac{1}{2}x - \frac{1}{12}x^3\right) + \frac{1}{2}\left(-\frac{1}{2}x - \frac{1}{12}x^3\right)^2 + \frac{1}{6}\left(-\frac{1}{2}x - \frac{1}{12}x^3\right)^3$$

$$= 1 + \left(-\frac{1}{2}x - \frac{1}{12}x^3\right) + \frac{1}{2}\left(-\frac{1}{2}x\right)^2 + \frac{1}{6}\left(-\frac{1}{2}x\right)^3$$

$$= 1 + \left(-\frac{1}{2}x - \frac{1}{12}x^3\right) + \frac{1}{2}\left(-\frac{1}{2}x\right)^2 + \frac{1}{6}\left(-\frac{1}{2}x\right)^3$$

So

$$(\cos x)^{\frac{1}{x}} = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{5}{48}x^3 + o(x^3).$$

<u>Exercise 4</u>

Reminder 4

4.1 Limited Development of order n in a neighborhood of x_0

We say that f admits a limited Development of order n in a neighborhood of x_0 if and only if the function $F: h \longrightarrow F(h) = f(h + x_0)$ admits a limited Development of order n in a neighborhood of 0.And if

$$F(h) = \sum_{k=0}^{n} h^k + h^n \varepsilon_1(h) \text{ with } \lim_{h\to 0} \varepsilon_1(h) = 0.$$

Then for all $x \in v - \{x_0\}$

$$f(x) = \sum_{k=0}^{n} a_n (x - x_0)^n + (x - x_0)^n \varepsilon(x) \text{ with } \lim_{x\to x_0} \varepsilon(x) = 0.$$

4.2 Limited Development of order n in a neighborhood of ∞

We say that f admits a limited Development of order n in a neighborhood of $+\infty$ ($-\infty$, respectively) if and only if the function $F: h \longrightarrow F(h) = f(\frac{1}{h})$ admits a limited Development of order n in a neighborhood of 0.And if

$$F(h) = \sum_{k=0}^{n} h^k + h^n \varepsilon_1(h) \text{ with } \lim_{\substack{> \\ x\to 0}} \varepsilon_1(h) = 0 \left(\lim_{\substack{< \\ x\to 0}} \varepsilon_1(h) = 0 \text{, respectively}\right).$$

Then

$$f(x) = \sum_{k=0}^{n} \frac{1}{x^k} + \frac{1}{x^n}\varepsilon(x) \text{ with } \lim_{x\to+\infty} \varepsilon(x) = 0 \left(\lim_{x\to-\infty} \varepsilon(x) = 0, \text{ respectively}\right) \text{ where } \varepsilon(x) = \varepsilon_1\left(\frac{1}{x}\right).$$

4.3 Study of infinite branches of curves

To study the infinite branches and determine the asymptotic lines of the graph (C_f) of function f in the neighborhood of $+\infty$ ($-\infty$, respectively), we develop the function f in the neighborhood of $+\infty$ ($-\infty$, respectively) to the smallest order n, where $a_n \neq 0$ and $n \in \mathbb{N}^*$.

1) Find a limited Development of order n in a neighborhood of x_0 for the functions f in each of the following cases.

1°) $n = 4\,,\; x_0 = 1\,, f(x) = \frac{\ln x}{x^2}$.

$$F(h) = f(x_0 + h) = f(1 + h) = \frac{\ln(1+h)}{(1+h)^2} = \frac{h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4}{1 + 2h + h^2}$$

By Euclidean dividing of $h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4$ over $1 + 2h + h^2$, according to increasing powers of x, keeping only terms with degrees less than or equal to 4, we get

$$\frac{\ln(1+h)}{(1+h)^2} = h - \frac{5}{2}h^2 + \frac{13}{3}h^3 - \frac{77}{12}h^4 + o(h^4).$$

By putting $h = x - 1$ we get

$$\frac{\ln x}{x^2} = x - 1 - \frac{5}{2}(x-1)^2 + \frac{13}{3}(x-1)^3 - \frac{77}{12}(x-1)^4 + o((x-1)^4).$$

4°) $n = 1\;\;, x_0 = +\infty\,, f(x) = \sqrt[3]{x^3 + x^2}$.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h}\sqrt[3]{1+h} = \frac{1}{h}\left(1 + \frac{1}{3}h - \frac{1}{9}h^2\right) = \frac{1}{h} + \frac{1}{3} - \frac{1}{9}h + o(h)$$

Substitution $h = \frac{1}{x}$ we get

$$f(x) = \sqrt[3]{x^3 + x^2} = \frac{1}{3} + x - \frac{1}{9}\frac{1}{x} + o\left(\frac{1}{x}\right)$$

5°) $n = 1\;\;,\; x_0 = +\infty\,, f(x) = x^2 \ln\left(\frac{xe^{\frac{1}{x}} + 1}{\mathrm{x}}\right)$.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h^2}\ln\left(h + e^h\right).$$

We have

$$h + e^h = 1 + 2h + \frac{1}{2}h^2 + \frac{1}{6}h^3.$$

So

$$\begin{aligned}
F(h) &= \frac{1}{h^2}\ln\left(h + e^h\right) = \frac{1}{h^2}\ln\left(1 + \left(h + e^h - 1\right)\right) \\
&= \frac{1}{h^2}\left(2h + \frac{1}{2}h^2 + \frac{1}{6}h^3 - \frac{1}{2}\left(2h + \frac{1}{2}h^2 + \frac{1}{6}h^3\right)^2 + \frac{1}{3}\left(2h + \frac{1}{2}h^2 + \frac{1}{6}h^3\right)^3\right) \\
&= \frac{1}{h^2}\left(2h + \frac{1}{2}h^2 + \frac{1}{6}h^3 - \frac{1}{2}\left(2h + \frac{1}{2}h^2\right)^2 + \frac{1}{3}(2h)^3\right) \\
&= \frac{1}{h^2}\left(2h + \frac{1}{2}h^2 + \frac{1}{6}h^3 - \frac{1}{2}\left((2h)^2 + 4h\frac{1}{2}h^2\right) + \frac{1}{3}(2h)^3\right)
\end{aligned}$$

So

$$F(h) = \frac{2}{h} - \frac{3}{2} + \frac{11}{6}h + o(h).$$

Substitution $h = \frac{1}{x}$ we get

$$f(x) = 2x - \frac{3}{2} + \frac{11}{6x} + o\left(\frac{1}{x}\right).$$

2) In questions, 4°, 5°, show that the graph (C_f) accepts a slanting asymptote that requires an equation, and then determine its relative position in the neighborhood of ∞.

4°) $f(x) = \sqrt[3]{x^3 + x^2}$. We have $f(x) = \frac{1}{3} + x - \frac{1}{9x} + o\left(\frac{1}{x}\right)$.

So

$$\lim_{x \to +\infty} \left[f(x) - \left(\frac{1}{3} + x\right)\right] = \lim_{x \to +\infty} \left[-\frac{1}{9x} + \frac{1}{x}\varepsilon(x)\right] = 0.$$

(C_f) accepts an asymptotic line (Δ) in a neighborhood of $+\infty$, which has an equation of the form $y = x + \frac{1}{3}$.

$$f(x) - \left(\frac{1}{3} + x\right) = -\frac{1}{9x} + \frac{1}{x}\varepsilon(x).$$

in a neighborhood of $+\infty$ then $f(x) - \left(\frac{1}{3} + x\right) = -\frac{1}{9x} + \frac{1}{x}\varepsilon(x) < 0$ so (C_f) is located under the asymptotic line (Δ).

5°) $f(x) = x^2 \ln\left(\frac{xe^{\frac{1}{x}}+1}{x}\right)$. We have $f(x) = 2x - \frac{3}{2} + \frac{11}{6x} + o\left(\frac{1}{x}\right)$.

So

$$\lim_{x \to +\infty} \left[f(x) - \left(2x - \frac{3}{2}\right)\right] = \lim_{x \to +\infty} \left[\frac{11}{6x} + o\left(\frac{1}{x}\right)\right] = 0.$$

(C_f) accepts an asymptotic line (Δ) in a neighborhood of $+\infty$, which has an equation of the form $2x - \frac{3}{2}$.

$$f(x) - \left(2x - \frac{3}{2}\right) = \frac{11}{6x} + o\left(\frac{1}{x}\right).$$

in a neighborhood of $+\infty$ then $f(x) - \left(\frac{1}{3} + x\right) = -\frac{1}{9x} + \frac{1}{x}\varepsilon(x) > 0$ so (C_f) is located above the asymptotic line (Δ).

<u>Exercise 5</u>

Reminder 5

5. 1 Calculation of limits

When calculating the limit $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ and if we obtain one of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$‘, to remove the indeterminacy we develop the functions f and g in the neighborhood of x_0 to the smallest orders m and n, respectively, where $b_n \neq 0$ and $a_n \neq 0$.

Using limited development, calculate the following limits.

1°) $\lim_{x\to 0}\frac{x^2\cos x-(e^x-1)^2}{\sin^3 x}$. We have

$$x^2\cos x-(e^x-1)^2 = x^2(1)-\left(x+\frac{1}{2}x^2\right)^2 = -x^3+x^3\varepsilon_1(x) \quad \text{and} \quad \sin^3 x = x^3+x^3\varepsilon_0(x).$$

So

$$\lim_{x\to 0}\frac{x^2\cos x-(e^x-1)^2}{\sin^3 x} = \lim_{x\to 0}\frac{-x^3+x^3\varepsilon_1(x)}{x^3+x^3\varepsilon_0(x)} = \lim_{x\to 0}\frac{-1+\varepsilon_1(x)}{1+\varepsilon_0(x)} = -1.$$

3°) $\lim_{x\to 1}\frac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-\sqrt[4]{x^3}}$. We put $x=h+1$, so $F(h)=f(1+h)=\frac{\sqrt[3]{1+h}-\sqrt{1-2h-6h^2-4h^3-h^4}}{\sqrt[4]{1+3h+3h^2+h^3}-1}$.

$$\sqrt[3]{1+h}-\sqrt{1-2h-6h^2-4h^3-h^4} = \sqrt[3]{1+h}-\sqrt{1-2h}$$

$$= \left(1+\frac{1}{3}h\right)-(1-h) = \frac{4}{3}h+o(h).$$

We have also

$$\sqrt[4]{1+3h+3h^2+h^3}-1 = 1+\frac{3}{4}h-1 = \frac{3}{4}h+o(h).$$

So

$$\lim_{x\to 1}\frac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-\sqrt[4]{x^3}} = \lim_{h\to 0}\frac{\frac{4}{3}h+h\varepsilon_1(h)}{\frac{3}{4}h+h\varepsilon_0(h)} = \lim_{h\to 0}\frac{\frac{4}{3}+\varepsilon_1(h)}{\frac{3}{4}+\varepsilon_0(h)} = \frac{16}{9}.$$

5°) $\lim_{x\to +\infty} x^{\frac{3}{2}}\left(\sqrt{x-1}+\sqrt{x+1}-2\sqrt{x}\right)$.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h^2}\left(\sqrt{1-h}+\sqrt{1+h}-2\right) = \frac{1}{h^2}\left(-\frac{1}{4}h^2+h^2\varepsilon(h)\right) = -\frac{1}{4}+\varepsilon(h).$$

Substitution $h=\frac{1}{x}$ we get

$$x^{\frac{3}{2}}\left(\sqrt{x-1}+\sqrt{x+1}-2\sqrt{x}\right) = -\frac{1}{4}+\varepsilon'(x) \quad \text{with} \quad \lim_{x\to +\infty}\varepsilon'(x)=0.$$

So

$$\lim_{x\to+\infty} x^{\frac{3}{2}}\left(\sqrt{x-1}+\sqrt{x+1}-2\sqrt{x}\right)=-\frac{1}{4}.$$

Exercise 6

Reminder 6

6.1 Study the relative position of the graph and the tangent line

To determine the relative position of the graph of a function f and its tangent line at the point x_0, we develop the function f in the neighborhood of x_0 to the smallest order n such that $a_n \neq 0$ and $n \geq 2$.

Deduce the equation of the tangent (T) to the curve $\left(C_f\right)$ at the abscissa point $x=0$, and determine the relative positions of $\left(C_f\right)$ and (T).

1°) $f(x)=\begin{cases}\frac{1}{x}-\frac{1}{x^2}\arctan\left(x+\frac{8}{15}x^3\right), & x\neq 0\\ 0 & , x=0\end{cases}.$

We have

$$\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5+o(x^5).$$

So for $x \neq 0$ then

$$\arctan\left(x+\frac{8}{15}x^3\right)=\left(x+\frac{8}{15}x^3\right)-\frac{1}{3}\left(x+\frac{8}{15}x^3\right)^3+\frac{1}{5}(x)^5$$
$$=x+\frac{1}{5}x^3-\frac{1}{3}x^5+o(x^5).$$

So

$$\frac{1}{x}-\frac{1}{x^2}\arctan\left(x+\frac{8}{15}x^3\right)=-\frac{1}{5}x+\frac{1}{3}x^3+o(x^3).$$

Since $f(0)=0=a_0$ so f is differentiable at $x_0=0$, and therefore $\left(C_f\right)$ accepts tangent line wich we denote by (T). And the equation of (T) is $y=-\frac{1}{5}x$.

We have $f(x)-\left(-\frac{1}{5}x\right)=\frac{1}{3}x^3+o\left(x^3\right)=x^3\left(\frac{1}{3}+\varepsilon(x)\right)$.

If x is sufficiently close to 0, the sign of the difference $f(x)-\left(-\frac{1}{5}x\right)$ is the same sign of $\frac{1}{3}x^3$, hence the following result:

For $x<0$, $\left(C_f\right)$ is located under the tangent and for $x>0$, $\left(C_f\right)$ is located above the tangent.

We conclude that $\left(C_f\right)$ accepts an inflection point $A_0(0,0)$.

3°) $f(x)=\frac{1}{1+\ln(x+\cos x)}$.

f defined, continuous, and differentiable at 0, so let us develop f at 0 to order 2.

We have

$$x+\cos x = 1 + x - \frac{1}{2}x^2 + o(x^2) \quad \text{and } \ln(1+h) = h - \frac{1}{2}h^2 + o(h^2)$$

So

$$\ln(1+h) = 1 + \ln\big(1 + (x+\cos x - 1)\big) = 1 + \left(x - \frac{1}{2}x^2\right) - \frac{1}{2}\left(x - \frac{1}{2}x^2\right)^2$$

$$= 1 + \left(x - \frac{1}{2}x^2\right) - \frac{1}{2}(x)^2$$

$$= 1 + \left(x - \frac{1}{2}x^2\right) - \frac{1}{2}(x)^2$$

$$= 1 + x - x^2.$$

By Euclidean dividing of 1 over $x - x^2$, according to increasing powers of x, keeping only terms with degrees less than or equal to 2, we get

1	$1+x-x^2$
$1+x-x^2$	$1-x+2x^2$
$-x+x^2$	
$-x-x^2$	
$2x^2$	
$2x^2$	
0	

So

$$f(x) = \frac{1}{1+\ln(x+\cos x)} = 1 - x + 2x^2 + o(x^2).$$

And (C_f) accepts tangent line wich we denote by (T). And the equation of (T) is $y = 1 - x$.

We have $f(x) - (1-x) = 2x^2 + o(x^2) = x^2(2 + \varepsilon(x))$.

If x is sufficiently close to 0, the sign of the difference $f(x) - (1-x)$ is the same sign of $2x^2$, hence the following result:

For $x < 0$ or $x > 0$, (C_f) is located above the tangent.

__Exercise 6__

In each of the following cases, show that the curve (C_f) of the function f accepts asymptote (Δ) in the vicinity of ∞, which requires an equation for it and examining the relative position of (C_f) and (Δ).

1°) $f(x) = x^2\sqrt{\frac{x-1}{x^3+2x}}$.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h^2}\sqrt{\frac{(-h+1)h^2}{1+2h^2}} = \frac{1}{h}\sqrt{1+\frac{1-h}{1+2h^2}-1} = \frac{1}{h}\sqrt{1-\frac{h+2h^2}{1+2h^2}}$$

Let us develop f at $+\infty$ to order 1.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h^2}\sqrt{\frac{(-h+1)h^2}{1+2h^2}}$$

$$= \frac{1}{h}\sqrt{1+\frac{1-h}{1+2h^2}-1}$$

$$= \frac{1}{h}\sqrt{1-\frac{h+2h^2}{1+2h^2}}$$

By Euclidean dividing of 1 over $x - x^2$, according to increasing powers of x, keeping only terms with degrees less than or equal to 2, we get

$$\begin{array}{r|l} h+2h^2 & 1+2h^2 \\ \cline{2-2} h & h+2h^2 \\ \hline 2h^2 & \\ 2h^2 & \\ \hline 0 & \end{array}$$

So

$$\frac{h+2h^2}{1+2h^2} = h + 2h^2 + o(h^2).$$

We have $\sqrt{1-h} = 1 - \frac{1}{2}h - \frac{1}{8}h^2 + o(h^2)$, so

$$\frac{1}{h}\sqrt{1-\frac{h+2h^2}{1+2h^2}} = \frac{1}{h}\left(1-\frac{1}{2}(h+2h^2)-\frac{1}{8}(h+2h^2)^2\right)$$

$$= \frac{1}{h}\left(1-\frac{1}{2}(h+2h^2)-\frac{1}{8}(h)^2\right)$$

$$= \frac{1}{h}\left(1-\frac{1}{2}h-\frac{9}{8}h^2+o(h^2)\right)$$

$$= \frac{1}{h}-\frac{1}{2}-\frac{9}{8}h+o(h).$$

Substitution $h = \frac{1}{x}$ we get

$$f(x) = x - \frac{1}{2} - \frac{9}{8x} + o\left(\frac{1}{x}\right) = x - \frac{1}{2} - \frac{9}{8x} + \frac{1}{x}\varepsilon(x) \quad \text{with} \quad \lim_{x\to+\infty} \varepsilon(x) = 0.$$

(C_f) accepts an asymptotic line (Δ) in a neighborhood of $+\infty$, which has an equation of the form $y = x - \frac{1}{2}$.

$$f(x) - \left(x - \frac{1}{2}\right) = -\frac{9}{8x} + \frac{1}{x}\varepsilon(x).$$

in a neighborhood of $+\infty$ then $f(x) - \left(x - \frac{1}{2}\right) = -\frac{9}{8x} + \frac{1}{x}\varepsilon(x) < 0$ so (C_f) is located under the asymptotic line (Δ).

3°) $f(x) = x^2 \sin\frac{x-2}{x^2+x+1}$

let us develop f at ∞ to order 1.

$$F(h) = f\left(\frac{1}{h}\right) = \frac{1}{h^2}\sin\frac{h(1-2h)}{1+h+h^2}.$$

We have

$$\sin(t) = t - \frac{1}{6}t^3 + o(t^3) \quad \text{and} \quad \frac{h(1-2h)}{1+h+h^2} = h - 3h^2 + 2h^3 + o(h^3).$$

So

$$\begin{aligned} F(h) &= \frac{1}{h^2}\sin\frac{h(1-2h)}{1+h+h^2} \\ &= \frac{1}{h^2}\left[(h - 3h^2 + 2h^3) - \frac{1}{6}(h - 3h^2 + 2h^3)^3\right] \\ &= \frac{1}{h^2}\left[(h - 3h^2 + 2h^3) - \frac{1}{6}(h)^3\right] \\ &= \frac{1}{h} - 3 + \frac{11}{6}h + o(h) \end{aligned}$$

Substitution $h = \frac{1}{x}$ we get

$$f(x) = x - 3 + \frac{11}{6x} + o\left(\frac{1}{x}\right) = x - 3 + \frac{11}{6x} + + \frac{1}{x}\varepsilon(x) \quad \text{with} \quad \lim_{x\to\infty} \varepsilon(x) = 0.$$

(C_f) accepts an asymptotic line (Δ) in a neighborhood of ∞, which has an equation of the form $y = x - 3$.

$$f(x) - (x - 3) = \frac{11}{6x} + \frac{1}{x}\varepsilon(x).$$

in a neighborhood of ∞ then $f(x)-(x-3)=\frac{11}{6x}++\frac{1}{x}\varepsilon(x)>0$ so

in a neighborhood of $+\infty$ (C_f) is located above the asymptotic line (Δ),

in a neighborhood of $-\infty$ (C_f) is located under the asymptotic line (Δ).

Exercise 8 (Short answers)

1) $u(x)=1+\frac{1}{2}x-\frac{3}{8}x^2+x^2\varepsilon(x)$ and $v(x)=1+\frac{1}{2}x+\frac{11}{8}x^2+x^2\varepsilon(x)$.

2) $\lim\limits_{\substack{> \\ x\to 0}}\frac{f(x)-1}{x}=\lim\limits_{\substack{> \\ x\to 0}}\left(\frac{u(x)-1}{x}\right)=\frac{1}{2}$, $\lim\limits_{\substack{< \\ x\to 0}}\frac{f(x)-1}{x}=\lim\limits_{\substack{< \\ x\to 0}}\left(\frac{v(x)-1}{x}\right)=\frac{1}{2}$.

So f is differentiable at 0 and the graph (C_f) accepts a tangent (T) at the point (0,1), and has an equation of the form $y=\frac{1}{2}x+1$.

$x>0\Longrightarrow f(x)-y=-\frac{3}{8}x^2+x^2\varepsilon(x)<0$ on a neighborhood of 0,

$x<0\Longrightarrow f(x)-y=-\frac{3}{8}x^2+x^2\varepsilon(x)>0$ on a neighborhood of 0.

For $x>0$, (C_f) is located under the tangent.
For $x<0$, (C_f) is located above the tangent.

We conclude that (C_f) accepts an inflection point A(0,1).

3) $u(x)=x+\frac{1}{2}-\frac{3}{8}\frac{1}{x}+\frac{1}{x}\varepsilon(x)$ where $\lim\limits_{x\to+\infty}\varepsilon(x)=0$.

$v(x)=-\frac{1}{4}-2x+\frac{49}{64}\frac{1}{x}+\frac{1}{x}\varepsilon'(x)$ where $\lim\limits_{x\to-\infty}\varepsilon'(x)=0$.

4) (C_f) accepts an asymptotic line (Δ) in a neighborhood of $+\infty$, which has an equation of the form $y=x+\frac{1}{2}$, and an asymptotic line (Δ') in a neighborhood of $-\infty$, which has an equation of the form $y=-2x-\frac{1}{4}$.

in a neighborhood of $+\infty$ then $f(x)-\left(x+\frac{1}{2}\right)=-\frac{3}{8}\frac{1}{x}+\frac{1}{x}\varepsilon(x)<0$ so (C_f) is located under the asymptotic line (Δ).

in a neighborhood of $-\infty$ then $f(x)-\left(-\frac{1}{4}-2x\right)=\frac{49}{64}\frac{1}{x}+\frac{1}{x}\varepsilon'(x)<0$ so (C_f) is located under the asymptotic line (Δ').