Exercise series No.3

Note: ***questions marked * left to the students***

Exercise 01

Using the Tyler-Lagrange formula, find a limited development of order $n \in \mathbb{N}^*$ at point x_0, of the function f in each of the following cases.

a) $x_0 = 1\ ;\ f(x) = \frac{1}{x}$, b) $x_0 = 0\ ;\ f(x) = xe^{2x}$, c) $x_0 = 2\ ;\ f(x) = \frac{x^2-2x-1}{x^2-4x+3}$

d)* $x_0 = 0\ ;\ f(x) = \sin 2x$, e)* $x_0 = 3\ ;\ f(x) = \ln\frac{x-2}{5-x}$.

Exercise 02

1) Using Tyler's formula, prove the following

a) $\forall x \in \mathbb{R}_+ : x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$,

b) $\forall x \in \left[0, \frac{\pi}{2}\right] : x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}$.,

d)* $\forall x \in \mathbb{R}_+ : 1 + \frac{x}{3} - \frac{x^2}{9} \leq \sqrt[3]{1+x} \leq 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5}{81}x^3$.

2) Determine the local Extrema of f in each of the following cases.

a)* $f(x) = x^3 - 2ax^2 + a^2x\ (a > 0)$, b) $f(x) = x^2(a-x)^2\ (a \in \mathbb{R})$.

Exercise 03

Find a limited Development of order n in a neighborhood of 0 for the functions f in each of the following cases

a) $n = 6$, $f(x) = \frac{x^2+x-1}{x^2+2}$, b) $n = 3$, $f(x) = e^x\sqrt{1-x}$

c)* $n = 4$, $f(x) = \cos x . \ln(1+x)$, d) $n = 4$, $f(x) = \ln\left(x + \sqrt{\cos x}\right)$

e)* $n = 3$, $f(x) = e^{\cos x}$ f) $n = 3$, $f(x) = (\cos x)^{\frac{1}{x}}$.

Exercise 04

Find a limited Development of order n in a neighborhood of x_0 for the functions f in each of the following cases

1°) $n = 4$, $x_0 = 1$, $f(x) = \frac{\ln x}{x^2}$ 2°)* $n = 6$, $x_0 = -1$, $f(x) = \frac{x^2-1}{x^2+2x}$.

3°)* $n = 1$, $x_0 = \infty$, $f(x) = \sqrt{x^2-4x+5}$ 4°) $n = 1$, $x_0 = +\infty$, $f(x) = \sqrt[3]{x^3+x^2}$.

5°) $n = 1$, $x_0 = +\infty$, $f(x) = x^2 \ln\left(\frac{xe^{\frac{1}{x}}+1}{x}\right)$ 6°)* $n = 1$, $x_0 = \infty$, $f(x) = x^2 \sin\frac{x-1}{x^2+1}$.

2) In questions 3°, 4°, 5°, 6°, show that the graph (C_f) accepts a slanting asymptote that requires an equation, and then determine its relative position in the neighborhood of ∞.

Exercise 05

Using limited development, calculate the following limits.

1°) $\lim\limits_{x\to 0}\frac{x^2\cos x-(e^x-1)^2}{\sin^3 x}$ 2°) $\lim\limits_{x\to 0}\frac{xe^x-\sin x}{\ln(1+x)+x\tan x-x}$ 3°) $\lim\limits_{x\to 1}\frac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-\sqrt[4]{x^3}}$

4°)* $\lim\limits_{x\to 1}\frac{\sqrt[5]{2x-1}-1}{\sqrt[4]{x}-\sqrt{2-x}}$ 5°) $\lim\limits_{x\to +\infty} x^{\frac{3}{2}}\left(\sqrt{x+1}+\sqrt{x+1}-2\sqrt{x}\right)$

6°) $\lim\limits_{x\to +\infty} x^5\left(\operatorname{argsinh}\frac{1}{x}+\arcsin\frac{1}{x}-\frac{2}{x}\right)$.

Exercise 06

Deduce the equation of the tangent (T) to the curve (C_f) at the abscissa point $x = 0$, and determine the relative positions of (C_f) and (T).

1°) $f(x) = \begin{cases}\frac{1}{x}-\frac{1}{x^2}\arctan\left(x+\frac{8}{15}x^3\right), & x \neq 0\\ 0 & , x = 0\end{cases}$ 2°)* $f(x) = \begin{cases} e^{\frac{1}{x}\ln\frac{\cosh x}{\cos x}}, & x \neq 0\\ 1 & , x = 0\end{cases}$

3°) $f(x) = \frac{1}{1+\ln(x+\cos x)}$ 4°)* $f(x) = \begin{cases}\frac{\sqrt{1+2\sin x}-e^x}{\ln(1+x+x^2)-x}, & x \neq 0\\ -\frac{2}{3}, & x = 0\end{cases}$.

Exercise 07

In each of the following cases, show that the curve (C_f) of the function f accepts asymptote (Δ) in the vicinity of ∞, which requires an equation for it and examining the relative position of (C_f) and (Δ).

1°) $f(x) = x^2\sqrt{\frac{x-1}{x^3+2x}}$ 2°)* $f(x) = x^2 ln\left(\frac{x+2}{x+1}\right)$ 3°) $f(x) = x^2\sin\frac{x-2}{x^2+x+1}$

4°)* $f(x) = e^{\frac{1}{x}}\sqrt{x^2+2x}$.

Exercise 8

Let f be a function defined on $\mathbb{R}$ by $f(x) = \begin{cases}(x+1)e^{\frac{-x}{2x^2+2}} & , x > 0\\ \sqrt{4x^2+x+1}\cos\frac{x}{x^2+1}, & x \leq 0\end{cases}$, and let (C_f) the graph representing the function f. We put $u(x) = (x+1)e^{\frac{-x}{2x^2+2}}$ and

$$u(x) = (x+1)e^{\frac{-x}{2x^2+2}} \quad \text{and} \quad v(x) = \sqrt{4x^2+x+1}\cos\frac{x}{x^2+1}$$

1) Find a limited Development of order 2 in a neighborhood of 0 for the functions u and v.

2) Using the previous limited developments, show that the function f is differentiable at the point $x_0 = 0$ and deduce the relative position of (C_f) and (Δ). What do you conclude?

3) Find a limited Development of order 1 in a neighborhood of $+\infty$ for the function u and in a neighborhood of $-\infty$ for the function v.

4) Deduce that the graph (C_f) accepts two asymptotes in the neighborhood of $+\infty$ and $-\infty$, determining an equation for each, and then determining the relative position of each with respect to the curve (C_f).