Chapter 3

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3 Taylor's Formulas, Limited Development

3.1 Taylor's Formulas

3.1.1 Taylor's formula with Lagrange remainder

Theorem 3.1

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function of class C^n on the interval $[a, b]$ and $f^{(n)}$ is differentiatiable over the interval $]a, b[$ then:

For every two numbers $x, x_0$of the interval $[a, b]$ where $x \neq x_0$ we have:

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Where c is a real number from $]x_0, x[$ (or $[x, x_0], if x < x_0$). The remainder $\boldsymbol{R}_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ called Lagrange remainder.

Proof

Let us define the functions g and φ on the interval $[a, b]$ by

$$g(t) = f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(t)}{j!}(x - t)^j,$$

and

$$\varphi(t) = g(t) - \frac{g(x_0)}{(x_0 - x)^{n+1}}(t - x)^{n+1}$$

We have

$$g(x) = 0 \;\; ; \;\; g(x_0) = f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \text{ and } g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n.$$

We have too

$$\varphi(x) = \varphi(x_0) = 0$$

By applying Rolle's theorem to the function φ in the interval $[x_0, x]$ (or $[x, x_0], if x < x_0$) we get:

there is a real number c in $]x_0, x[$ (or $]x, x_0[, if x < x_0$) where

$$\varphi'(c) = 0$$

or

$$g'(c) - (n+1)\frac{g(x_0)}{(x_0 - x)^{n+1}}(c - x)^n = 0$$

or

$$-\frac{f^{(n+1)}(c)}{n!}(x - c)^n - (n+1)\frac{g(x_0)}{(x_0 - x)^{n+1}}(c - x)^n = 0$$

or

$$g(x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

So

$$f(x) - \sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Finally we obtain

$$f(x) = \sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

3.1.2 Taylor's formula with Young's remainder

Definition 3.1

Let f and g be functions defined in the neighborhood v of the point x_0 where

$\forall x \in v - \{x_0\}$: $g(x) \neq 0$, we say that f is negligible before g when $x \to x_0$ and we write $f = \mathrm{o}(g)$, if the following is true: $\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0$.

Definition 3.2

By putting $\varepsilon(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)$. The Taylor's- Lagrange formula is rewritten in the form:

$$f(x)=\sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j+(x-x_0)^n\,\varepsilon(x)\quad \text{where}\ \lim_{x\to x_0}\varepsilon(x)=0.$$

The remainder $\boldsymbol{R}_n=(x-x_0)^n\,\varepsilon(x)$, is called Young remainder.

And by putting $\boldsymbol{R}_n=(x-x_0)^n\varepsilon(x)=\mathrm{o}((x-x_0)^n)$, the previous formula is rewritten into the form:

$$f(x)=\sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j+\mathrm{o}((x-x_0)^n).$$

Note: We mention that $\boldsymbol{R}_n=\mathrm{o}((x-x_0)^n)$ means that $\lim_{x\to x_0}\frac{\boldsymbol{R}_n}{(x-x_0)^n}=0$.

3.1.3 McLaurin's formula with Young's remainder

By taking $x_0=0$ in Taylor's-Young's formula, we get:

$$f(x)=\sum_{j=0}^{n}\frac{f^{(j)}(0)}{j!}x^j+o(x^n),$$

called McLaurin's- Young's formula.

Examples 3.1

1) Let $f(x)=e^x$; $x_0=0$ we have $f\in C^{\infty}(\mathbb{R})$; $\forall k\in\mathbb{N}^*: f^{(k)}(x)=e^x$ and

$$\forall k\in\mathbb{N}^*: f^{(k)}(0)=1.$$

So

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+R_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\dots+\frac{x^n}{n!}+R_n(x),$$

where

* $R_n(x)=\frac{x^{n+1}}{(n+1)!}e^c$ with $c\in]0,x[$ or $c\in]x,0[$ In Taylor's- Lagrange formula.

* $R_n(x)=x^n\varepsilon(x)=0(x^n)$ with $\lim_{x\to x_0}\varepsilon(x)=0$. In Taylor's - Young's formula.

2) Let $f(x)=\frac{1}{1-x}$; $x,x_0\in]1,+\infty[$ we have $f\in C^{\infty}(]1,+\infty[\ )$ and

$$\forall k\in\mathbb{N}: f^{(k)}(x_0)=\frac{k!}{(1-x_0)^{k+1}}.$$

So

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} x^k + R_n(x, x_0) = \sum_{k=0}^{n} \frac{1}{(1 - x_0)^{k+1}} x^k + R_n(x, x_0)$$

$$= \frac{1}{1 - x_0} + \frac{x - x_0}{(1 - x_0)^2} + \frac{(x - x_0)^2}{(1 - x_0)^3} + \frac{(x - x_0)^3}{(1 - x_0)^4} + \cdots . + \frac{(x - x_0)^n}{(1 - x_0)^n} + R_n(x, x_0).$$

Where

* $R_n(x, x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = \frac{(x - x_0)^{n+1}}{(1-c)^{n+2}}$ with $c \in]x_0, x[$ or $c \in]x, x_0[$ In Taylor's-Lagrange formula.

* $R_n(x, x_0) = (x - x_0)^n \varepsilon(x, x_0) = 0((x - x_0)^n)$ with $\lim_{x \to x_0} \varepsilon(x, x_0) = 0$. In Taylor's - Young's formula.

For example if $x_0 = 3$, then

$$f(x) = -\frac{1}{2} + \frac{1}{4}(x - 3) - \frac{1}{8}(x - 3)^2 + \frac{1}{16}(x - 3)^3 + \cdots . + \frac{(-1)^n}{2^n}(x - 3)^n + R_n(x).$$

$$R_n(x) = \frac{(x - 3)^{n+1}}{(1 - c)^{n+2}} \text{ with } c \in]3, x[\text{ or } c \in]x, 3[\text{ (Or } |c - 3| < |x - 3|),$$

or

$$R_n(x) = (x - 3)^n \varepsilon(x) = 0((x - 3)^n) \text{ with } \lim_{x \to 3} \varepsilon(x) = 0$$

3.1.4 Apply Tyler's formulas to find local extrema

Theorem 3.3

Let f be a function of class C^n in the neighborhood of the point x_0 such that:

$f'(x_0) = f''(x_0) = f^{(3)}(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Then

f accepts a local maximum (local minimum, respectively) at x_0 if and only if n is even and $f^{(n)}(x_0) < 0$ ($f^{(n)}(x_0) > 0$, respectively).

Proof

Applying the Tyler-Lagrange formula and taking into account the hypothesis on the derivatives we get:

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n \quad \text{where c is a real number between } x_0 \text{ and } x.$$

Assume that $f^{(n)}(x_0) < 0$.

Since $f^{(n)}$ is continuous at x_0, there exists a real number $\delta > 0$ such that:

$\forall x \in]x_0-\delta, x_0+\delta[: f^{(n)}(x) < 0 \Longrightarrow f^{(n)}(c) < 0$ (Because c confined between x_0, x).

If n is even, then $\dfrac{f^{(n)}(c)}{n!}(x - x_0)^n < 0.$ So $\forall x \in]x_0-\delta, x_0+\delta[: f(x) < f(x_0)$.

So $f(x_0)$ is local maximum.

In the same way the proof is performed if $f^{(n)}(x_0) > 0$.

If n is odd, then the sign of the difference $f(x) - f(x_0)$ changes at x_0.

Example 3.2

Let $f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4$.

Necessary condition:

$$f'(x) = 0 \Leftrightarrow x = 0, x = 1$$

Sufficient condition:

$f'(0) = f''(0) = f^{(3)}(0) = 0$ and $f^{(4)}(0) = -6 < 0 \Rightarrow f(0) = 0$ is local maximum value of f.

Similarly we have

$f'(1) = 0$ and $f''(1) = 1 > 0 \Rightarrow f(1) = -\frac{1}{20}$ is local minimum value of f.

3.2 Limited Development

3.2.1 Limited Development of order n in a neighborhood of 0

Definition 3.3

Let f be a function defined in a neighborhood of 0 - with the possible exception of 0 - we say that f admits a limited Development of order n in a neighborhood of 0 if and only if there exists a neighborhood v of 0 and constant numbers $a_0, a_1, a_2, \dots\dots\dots, a_n$ where

$$\forall x \in v\, ; x \neq 0: f(x) = a_0 + a_1 x + a_2 x^2 \dots\dots\dots + a_n x^n + x^n \varepsilon(x) \text{ with } \lim_{x\to 0} \varepsilon(x) = 0.$$

Or

$$\forall x \in v\, ; x \neq 0: f(x) = \sum_{k=0}^{n} a_k x^k + x^n \varepsilon(x) \text{ with } \lim_{x\to 0} \varepsilon(x) = 0.$$

The polynomial $a_0 + a_1 x + a_2 x^2 \dots\dots\dots + a_n x^n$, is called the regular part and we denote it by $P_n(x)$.

The term $x^n\varepsilon(x)$ (or $o(x^n)$) is called the remainder or the complementary term, and it is symbolized by $R_n(x)$.

Theorem 3.4 (uniqueness)

If a function f admits a limited Development of order n in the neighborhood of 0, then this Development is unique.

Proof

Assume that for all $x \in v - \{0\}$:

$$f(x) = a_0 + a_1x + a_2x^2 \dots\dots\dots + a_nx^n + x^n\varepsilon_1(x) \text{ with } \lim_{x\to 0}\varepsilon_1(x) = 0,$$

and

$$f(x) = b_0 + b_1x + b_2x^2 \dots\dots\dots + b_nx^n + x^n\varepsilon_2(x) \text{ with } \lim_{x\to 0}\varepsilon_2(x) = 0.$$

So we have for all $x \in v - \{0\}$:

$$a_0 - b_0 + (a_1 - b_1)x + (a_2 - b_2)x^2 \dots\dots\dots + (a_n - b_n)x^n + x^n\big(\varepsilon_1(x) - \varepsilon_2(x)\big) = 0.$$

Taking the limit at 0 we obtain

$$a_0 - b_0 = 0.$$

So we have for all $x \in v - \{0\}$:

$$(a_1 - b_1)x + (a_2 - b_2)x^2 \dots\dots\dots + (a_n - b_n)x^n + x^n\big(\varepsilon_2(x) - \varepsilon_1(x)\big) = 0,$$

or

$$(a_1 - b_1) + (a_2 - b_2)x \dots\dots\dots + (a_n - b_n)x^{n-1} + x^{n-1}\big(\varepsilon_2(x) - \varepsilon_1(x)\big) = 0.$$

Taking the limit again at 0 we obtain

$$(a_1 - b_1) = 0.$$

By continuing the operation, we obtain for all $x \in v - \{0\}$

$$(a_n - b_n) + \big(\varepsilon_2(x) - \varepsilon_1(x)\big) = 0,$$

from where

$$(a_n - b_n) = \lim_{x\to 0}\big(\varepsilon_1(x) - \varepsilon_2(x)\big) = 0$$

So

$$a_n = b_n \text{ and } \varepsilon_1(x) = \varepsilon_2(x).$$

Theorem 3.5

If a function f admits a limited Development of order n ($n \geq 1$) in the neighborhood of 0 and if $f(0) = a_0$ then f is differentiable at 0 and we have $f'(0) = a_1$.

Proof

If we have $f(0) = a_0$ we can write

$$\frac{f(x) - f(0)}{x - 0} = a_1 + a_2 x \dots\dots\dots + a_n x^{n-1} + x^{n-1}\varepsilon_1(x) \text{ with } \lim_{x\to 0}\varepsilon_1(x) = 0,$$

so

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\left(a_1 + a_2 x \dots\dots\dots + a_n x^{n-1} + x^{n-1}\varepsilon_1(x)\right) = a_1.$$

Theorem 3.6

If f is of class C^n in the neighborhood of 0, then f admits a limited development to the neighborhood of 0, which is obtained in the McLaurin's- Young's formula, i.e.:

$$f(x) = \sum_{j=0}^{n}\frac{f^{(j)}(0)}{j!}x^j + o(x^n).$$

Proof

The theorem results from the application of the Maclaurin-Young formula and the uniqueness of limited development.

Theorem 3.7

If an even (respectively odd) function admits an limited development to the neighborhood of 0, then its regular part is even (respectively odd).

Proof

Let f be an even function admitting a limited Development of order n in the neighborhood of 0 by the form: $f(x) = a_0 + a_1 x + a_2 x^2 \dots\dots\dots + a_n x^n + x^n\varepsilon(x)$ with $\lim_{x\to 0}\varepsilon(x) = 0$.

So

$$\forall x \in v\,;x \neq 0: f(-x) = f(x),$$

$$\forall x \in v\,;x \neq 0: \sum_{k=0}^{n}(-1)^k a_k x^k + (-x)^n\varepsilon(-x) = \sum_{k=0}^{n} a_k x^k + x^n\varepsilon(x).$$

According to the theorem 3.4 we obtain:

$$\forall k \in \{0,1,2,\dots\dots\dots,n\}: a_k = (-1)^k a_k \text{ and } \varepsilon(x) = (-1)^n\varepsilon(-x).$$

So, if k is even number, then $a_k = 0$.

In the same way, the proof is done if f is odd.

Corollary 3.1

The limited development of f is therefore writes:

$$f(x) = a_0 + a_2x^2 + \dots\dots + a_{2n}x^{2n} + o(x^{2n}) \text{ If } f \text{ is even.}$$

$$f(x) = a_1 + a_3x^2 + \dots\dots + a_{2n+1}x^{2n+1} + o(x^{2n+1}) \text{ If } f \text{ is odd.}$$

Remark 3.1

The order of the limited development is determined by the degree of the remainder $o(x^n)$ and not the degree of the regular part.

Example 3.3

$f(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 + o(x^5)$ Is a limited development of order 5.

$g(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 - \frac{7}{13}x^4 + 2x^5 + o(x^3)$ Is a limited development of order 3.

Since $-\frac{7}{13}x^4 + 2x^5 = o(x^3)$ we get $g(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 + o(x^3)$

3.2.2 Operation on Limited Development

Theorem 3.8

Let f, g be two functions admitting limited developments to the same order n in the neighborhood of 0. We denote their regulars parts as $P_n(x), Q_n(x)$, respectively. That is

$$f(x) = P_n(x) + x^n\varepsilon_1(x) \quad ; \quad g(x) = Q_n(x) + x^n\varepsilon_2(x).$$

Then, the functions $f + g, fg, \frac{f}{g}$ (if $\lim_{x\to 0} g(x) \neq 0$), fog (if $\lim_{x\to 0} g(x) = 0$), admitting limited developments of order n in the neighborhood of 0 and we have:

1) $f(x) + g(x) = P_n(x) + Q_n(x) + x^n\varepsilon_3(x)$ with $\lim_{x\to 0} \varepsilon_3(x) = 0$.

2) $f(x)g(x) = A_n(x) + x^n\varepsilon_4(x)$ with $\lim_{x\to 0} \varepsilon_4(x) = 0$.

Where $A_n(x)$ is the polynomial we obtain by retaining in the multiplication $P_n(x)Q_n(x)$ only the terms with degrees less than or equal to n.

3) $\frac{f(x)}{g(x)} = B_n(x) + x^n\varepsilon_5(x)$ with $\lim_{x\to 0} \varepsilon_5(x) = 0$.

Where $B_n(x)$ is the polynomial we obtain by Euclidean division of $P_n(x)$ by $Q_n(x)$ according to increasing powers of x keeping only terms with degrees less than or equal to n.

4) $fog(x) = C(x) + x^n \varepsilon_6(x)$ with $\lim_{x\to 0} \varepsilon_6(x) = 0$.

Where $C_n(x)$ is the polynomial we obtain by retaining in the composite $P_n o\, Q_n(x)$ only the terms with degrees less than or equal to n.

Proof

Let us prove the third and fourth cases because the first and second cases are deduced by direct calculation.

a) Prove the third cases

We first recall with the following proposition

Proposition 3.1 (DIVISION BY INCREASING POWER ORDER)

Let $n, m, p \in \mathbb{N}^*$ with $n \neq 0$, and A, B two polynomials. We write them

$A(x) = a_0 + a_1 x + a_2 x^2 \dots\dots\dots + a_p x^p$ and $B(x) = b_0 + b_1 x + b_2 x^2 \dots\dots\dots + b_m x^m$.

We assume that $b_0 \neq 0$. Then there exist a unique pair (Q, R) of polynomials such that
$\begin{cases} A(x) = B(x)Q(x) + x^{n+1}R(x) \\ \deg(Q) \leq n \end{cases}$.

Let us now return to proving the third case in Theorem 3.6. we've got.

$$f(x) = P_n(x) + x^n \varepsilon_1(x),$$

$$g(x) = Q_n(x) + x^n \varepsilon_2(x).$$

The division according to the increasing powers to the order n of $P_n(x)$ by $Q_n(x)$ gives

$$\begin{cases} P_n(x) = Q_n(x)B_n(x) + x^{n+1}R(x) \\ \deg(B_n) \leq n \end{cases}.$$

From where

$$\begin{aligned} f(x) - x^n \varepsilon_1(x) &= P_n(x) \\ &= Q_n(x)B_n(x) + x^{n+1}R(x) \\ &= \big(g(x) - x^n \varepsilon_2(x)\big)B_n(x) + x^{n+1}R(x). \end{aligned}$$

So

$$f(x) = g(x)B_n(x) + x^n\big(\varepsilon_1(x) - \varepsilon_2(x)B_n(x) + xR(x)\big),$$

$$\frac{f(x)}{g(x)} = B_n(x) + x^n \frac{\left(\varepsilon_1(x) - \varepsilon_2(x)B_n(x) + xR(x)\right)}{g(x)}$$

Or

$$\frac{f(x)}{g(x)} = B_n(x) + x^n\varepsilon(x) \quad \text{with } \varepsilon(x) = \frac{\left(\varepsilon_1(x) - \varepsilon_2(x)B_n(x) + xR(x)\right)}{g(x)}.$$

Since $\lim_{x\to 0} g(x) \neq 0$ then $\lim_{x\to 0} \varepsilon(x) = 0$.

Finally we obtain

$$\frac{f(x)}{g(x)} = B_n(x) + x^n\varepsilon(x);\ \deg(B_n) \leq n \ \text{ with } \lim_{x\to 0} \varepsilon(x) = 0.$$

b) Prove the fourth cases

To prove the fourth case we need the following proposition.

Proposition 3.2

Let f be a functions admits limited development to the order n in the neighborhood of 0 where $f(x) = P_n(x) + x^n\varepsilon(x)$ with $\lim_{x\to 0} \varepsilon(x) = 0$ where $n \in \mathbb{N}^*$. Then for all $k \in \{1, 2, 3, \dots \dots \dots, n\}$: $f^k(x) = P_n^k(x) + x^n\varepsilon_1(x)$ with $\lim_{x\to 0} \varepsilon_1(x) = 0$.

Proof of proposition 3.2 (Proof by induction)

For $k = 1$: $f^1(x) = P_n^1(x) + x^n\varepsilon_1(x)$ is true (It suffices to take $\varepsilon_1 = \varepsilon$).

Assume that:

$$f^k(x) = P_n^k(x) + x^n\varepsilon_2(x),$$

and we prove that:

$$f^{k+1}(x) = P_n^{k+1}(x) + x^n\varepsilon_3(x).$$

Indeed

$$f^{k+1}(x) = f^k(x)f(x) = \left(P_n^k(x) + x^n\varepsilon_2(x)\right)\left(P_n(x) + x^n\varepsilon(x)\right)$$

$$= \left(P_n^k(x) + x^n\varepsilon_2(x)\right)\left(P_n(x) + x^n\varepsilon(x)\right)$$

$$= P_n^{k+1}(x) + x^nP_n(x)\varepsilon_2(x) + x^nP_n^k(x)\varepsilon(x) + x^{2n}\varepsilon_2(x)\varepsilon(x)$$

$$= P_n^{k+1}(x) + x^n\left(P_n(x)\varepsilon_2(x) + P_n^k(x)\varepsilon(x) + x^n\varepsilon_2(x)\varepsilon(x)\right).$$

By putting $\varepsilon_3(x) = P_n(x)\varepsilon_2(x) + P_n^k(x)\varepsilon(x) + x^n\varepsilon_2(x)\varepsilon(x)$, then $\lim_{x\to 0} \varepsilon_3(x) = 0$.

We return to proof the fourth cases

We have

$$f(x)=P_n(x)+x^n\varepsilon_1(x) \text{ where } P_n(x)=\sum_{k=0}^{n} b_k x^k \text{ and } \lim_{x\to 0}\varepsilon_1(x)=0,$$

and since $\lim_{x\to 0} g(x)=0$, then

$$g(x)=Q_n(x)+x^n\varepsilon_2(x) \text{ where } Q_n(x)=\sum_{k=1}^{n} a_k x^k = xR_n(x) \text{ and } \lim_{x\to 0}\varepsilon_2(x)=0.$$

So

$$fog(x)=P_n\big(Q_n(x)+x^n\varepsilon_2(x)\big)+\big(Q_n(x)+x^n\varepsilon_2(x)\big)^n\varepsilon_1\big(Q_n(x)+x^n\varepsilon_2(x)\big)$$

$$=P_n\big(Q_n(x)+x^n\varepsilon_2(x)\big)+\big(xR_n(x)+x^n\varepsilon_2(x)\big)^n\varepsilon_1\big(xR_n(x)+x^n\varepsilon_2(x)\big)$$

$$=P_n\big(Q_n(x)+x^n\varepsilon_2(x)\big)+x^n\big(R_n(x)+x^{n-1}\varepsilon_2(x)\big)^n\varepsilon_1\big(xR_n(x)+x^n\varepsilon_2(x)\big).$$

Since $\lim_{x\to 0}\big(xR_n(x)+x^n\varepsilon_2(x)\big)=0$, then we put

$$\big(R_n(x)+x^{n-1}\varepsilon_2(x)\big)^n\varepsilon_1\big(xR_n(x)+x^n\varepsilon_2(x)\big)=\varepsilon_3(x) \text{ with } \lim_{x\to 0}\varepsilon_3(x)=0$$

So

$$fog(x)=P_n\big(Q_n(x)+x^n\varepsilon_2(x)\big)+x^n\varepsilon_3(x)$$

$$=b_0+\sum_{k=1}^{n} b_k\big(Q_n(x)+x^n\varepsilon_2(x)\big)^k+x^n\varepsilon_3(x)$$

$$=b_0+\sum_{k=1}^{n} b_k\Big(Q_n^k(x)+x^n\varepsilon_4(x)\Big)+x^n\varepsilon_3(x) \quad \text{(According to proposition 3.2)}$$

$$=b_0+\sum_{k=1}^{n} b_k Q_n^k(x)+x^n\varepsilon_4(x)\sum_{k=1}^{n} b_k+x^n\varepsilon_3(x)$$

$$=b_0+\sum_{k=1}^{n} b_k Q_n^k(x)+x^n\left(\varepsilon_4(x)\sum_{k=1}^{n} b_k+\varepsilon_3(x)\right)$$

$$=P_n o Q_n(x)+x^n\varepsilon_5(x) \text{ with } \lim_{x\to 0}\varepsilon_5(x)=0.$$

Let $C_n(x)$ be a polynomial we obtain by retaining in the composite $P_n o\, Q_n(x)$ only the terms with degrees less than or equal to n.

Finally we obtain

$$fog(x) = C_n(x) + x^n\varepsilon_6(x); \deg(C_n) \leq n \text{ with } \lim_{x\to 0}\varepsilon_6(x) = 0.$$

Examples 3.4

1) $f(x) = \ln(1+x) = \underbrace{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}_{P_n(x)} + x^3\varepsilon_1(x)$ with $\lim\limits_{x\to 0}\varepsilon_1(x) = 0$

$g(x) = e^x = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}_{Q_n(x)} + x^3\varepsilon_2(x)$ with $\lim\limits_{x\to 0}\varepsilon_2(x) = 0$

* $f(x) + g(x) = \ln(1+x)e^x$

$$= P_n(x) + Q_n(x) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$$

$$= 1 + 2x + \frac{1}{2}x^3 + x^3\varepsilon_3(x) \text{ with } \lim_{x\to 0}\varepsilon_3(x) = 0.$$

* $f(x)g(x) = \ln(1+x) + e^x$

$$= P_n(x)Q_n(x) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$$

$$= x\left(1 + x + \frac{1}{2}x^2\right) - \frac{1}{2}x^2(1+x) + \frac{1}{3}x^3(1)$$

$$= x + x^2 + \frac{1}{2}x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^3$$

$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3\varepsilon_4(x) \text{ with } \lim_{x\to 0}\varepsilon_4(x) = 0.$$

2) $f(x) = \sinh x = \underbrace{x + \frac{1}{6}x^3}_{P_n(x)} + x^3\varepsilon_1(x)$ with $\lim\limits_{x\to 0}\varepsilon_1(x) = 0$

$g(x) = \sqrt{1+x} = \underbrace{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}_{Q_n(x)} + x^3\varepsilon_2(x)$ with $\lim\limits_{x\to 0}\varepsilon_2(x) = 0$

* We have $\lim\limits_{x\to 0} g(x) \neq 0$, so $\frac{f}{g}$ admits a limited development.

By Euclidean division of $P_n(x)$ by $Q_n(x)$ according to increasing powers of x we obtain

$$\begin{array}{r|l}
x + \frac{1}{6}x^3 & 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \\ \cline{2-2}
x + \frac{1}{2}x^2 - \frac{1}{8}x^3 & x - \frac{1}{2}x^2 + \frac{13}{24}x^3 \\ \cline{1-1}
-\frac{1}{2}x^2 + \frac{7}{24}x^3 &
\end{array}$$

$$\begin{array}{r|l} -\frac{1}{2}x^2-\frac{1}{4}x^3 & \\ \hline \frac{13}{24}x^3 & \\ \frac{13}{24}x^3 & \\ \hline 0 & \end{array}$$

So

$$\frac{f(x)}{g(x)}=\frac{\sinh x}{\sqrt{1+x}}=x-\frac{1}{2}x^2+\frac{13}{24}x^3+x^3\varepsilon_3(x) \text{ with } \lim_{x\to 0}\varepsilon_3(x)=0.$$

3) $f(x)=e^x=\underbrace{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}_{P_n(x)}+x^3\varepsilon_1(x)$ with $\lim_{x\to 0}\varepsilon_1(x)=0,$

$g(x)=\sin x=\underbrace{x-\frac{1}{6}x^3}_{Q_n(x)}+x^3\varepsilon_2(x)$ with $\lim_{x\to 0}\varepsilon_2(x)=0.$

* We have $\lim_{x\to 0}g(x)=0$, so fog admits a limited development.

$$fog(x)=e^{\sin x}=P_n o Q_n(x)$$

$$=1+\left(x-\frac{1}{6}x^3\right)+\frac{1}{2}\left(x-\frac{1}{6}x^3\right)^2+\frac{1}{6}\left(x-\frac{1}{6}x^3\right)^3$$

$$=1+\left(x-\frac{1}{6}x^3\right)+\frac{1}{2}(x)^2+\frac{1}{6}(x)^3$$

$$=1+x+\frac{1}{2}x^2+x^3\varepsilon_3(x) \text{ with } \lim_{x\to 0}\varepsilon_3(x)=0.$$

Theorem 3.9 (Integration of limited development.)

Let $f:[-a;\ a]\longrightarrow\mathbb{R}$ an integrable function and admitting in the neighborhood of 0 the limited development:

$$f(x)=a_0+a_1x+a_2x^2\dots\dots\dots+a_nx^n+x^n\varepsilon(x) \text{ with } \lim_{x\to 0}\varepsilon(x)=0,$$

Then the function $F: x\longrightarrow\int_0^x f(t)\,dt$ admits in the neighborhood of 0 the limited development of order $n+1$ following:

$$F(x)=a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3\dots\dots\dots+\frac{a_n}{n+1}x^{n+1}+x^{n+1}\varepsilon_1(x) \text{ with } \lim_{x\to 0}\varepsilon_1(x)=0,$$

i.e. the regular part of limited development of F is equal to the integral of regular part in limited development of f.

Proof

We have

$$F(x) = \int_0^x f(t)\,dt = \int_0^x (a_0 + a_1 t + \ a_2 t^2 \dots\dots\dots + a_n t^n)\,dt + \int_0^x t^n \varepsilon(t)\,dt$$

$$= \int_0^x (a_0 + a_1 t + \ a_2 t^2 \dots\dots\dots + a_n t^n)\,dt + \int_0^x t^n \varepsilon(t)\,dt.$$

$$= a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 \dots\dots\dots + \frac{a_n}{n+1} x^{n+1} + \int_0^x t^n \varepsilon(t)\,dt.$$

It is enough to show that

$$\int_0^x t^n \varepsilon(t)\,dt = o(x^{n+1}) \ .$$

Let $x > 0$. Since $\lim_{t \to 0} \varepsilon(t) = 0$, then the function ε is bounded in $]0, x]$ and $\forall t \in\ ]0, x]: \varepsilon(t) \leq |\varepsilon(t)| \leq \sup_{0<t\leq x} |\varepsilon(t)|$. So

$$\left| \int_0^x t^n \varepsilon(t)\,dt \right| \leq \int_0^x t^n |\varepsilon(t)|\,dt$$

$$\leq \int_0^x t^n \sup_{0<t\leq x} |\varepsilon(t)|\,dt = \sup_{0<t\leq x} |\varepsilon(t)| \int_0^x t^n\,dt = \sup_{0<t\leq x} |\varepsilon(t)| \frac{x^{n+1}}{n+1}.$$

So

$$\frac{\left| \int_0^x t^n \varepsilon(t)\,dt \right|}{x^{n+1}} \leq \frac{1}{n+1} \sup_{0<t\leq x} |\varepsilon(t)|$$

Since $\lim_{t \to 0} \varepsilon(t) = 0$, then $\lim_{x \to 0} \sup_{0<t\leq x} |\varepsilon(t)| = 0$ and from it

$$\lim_{x \to 0} \frac{\left| \int_0^x t^n \varepsilon(t)\,dt \right|}{x^{n+1}} = 0, \text{and from him } \lim_{x \to 0} \frac{\int_0^x t^n \varepsilon(t)\,dt}{x^{n+1}} = 0.$$

So

$$\int_0^x t^n \varepsilon(t)\,dt = o(x^{n+1}).$$

Corollary 3.1

If F is a primitive function of f over $[-a;\ a]$ then

$$F(x) = F(0) + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 \dots\dots\dots + \frac{a_n}{n+1} x^{n+1} + x^{n+1} \varepsilon_1(x) \text{ with } \lim_{x \to 0} \varepsilon_1(x) = 0.$$

Theorem 3.10 (Derivation of limited development.)

Let $f: [-a;\ a] \longrightarrow \mathbb{R}$ a differentiable function admitting a limited development of order n in the neighborhood of 0 If its derivative f' admits a limited development of order $n-1$ in the neighborhood of 0, then the regular part of the limited development of f' is the derivative of the regular part of the limited development of f.

Proof

The proof is based on the theorem 3.9 and the mean value theorem.

3.2.3 Limited Development of order n in a neighborhood of x_0

Definition 3.4

Let f be a function defined in a neighborhood of x_0 - with the possible exception of x_0- we say that f admits a limited Development of order n in a neighborhood of x_0 if and only if the function $F: h \longrightarrow F(h) = f(h + x_0)$ admits a limited Development of order n in a neighborhood of 0.And if

$$F(h) = a_0 + a_1 h + a_2 h^2 \dots\dots\dots \text{‘} a_n h^n + h^n \varepsilon_1(h) \text{ with } \lim_{h \to 0} \varepsilon_1(h) = 0.$$

Then for all $x \in v - \{x_0\}$:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \dots\dots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x, x_0) \text{ with}$$

$$\lim_{x \to x_0} \varepsilon(x, x_0) = 0.$$

3.2.4 Limited Development of order n in a neighborhood of ∞

Definition 3.5

Let f be a function defined in a neighborhood of $+\infty$ ($-\infty$, respectively). We say that f admits a limited Development of order n in a neighborhood of $+\infty$ ($-\infty$, respectively) if and only if the function $F: h \longrightarrow F(h) = f(\frac{1}{h})$ admits a limited Development of order n in a neighborhood of 0.And if

$$F(h) = a_0 + a_1 h + a_2 h^2 \dots\dots\dots \text{‘} a_n h^n + h^n \varepsilon_1(h) \text{ with } \lim_{\substack{> \\ x \to 0}} \varepsilon(h) = 0 \ \ (\lim_{\substack{< \\ x \to 0}} \varepsilon(h) = 0 \text{, respec)}$$

Then

$$f(x) = a_0 + a_1 \frac{1}{x} + a_2 \frac{1}{x^2} \dots\dots\dots + a_n \frac{1}{x^n} + \frac{1}{x^n} \varepsilon(x) \text{ with } \lim_{x \to +\infty} \varepsilon(x) = 0 \ \left(\lim_{x \to -\infty} \varepsilon(x) = 0, \text{ resp} \right),$$

where $\varepsilon(x) = \varepsilon_1\left(\frac{1}{x}\right)$.

Remark 3.2

If the function F admits a limited Development of order n in a neighborhood of 0, then the two limited Developments of f in the neighborhood of $+\infty$ and in the neighborhood of $-\infty$ are identical. In this case, we say that f admits a limited Development of order n in a neighborhood of infinity.

3.3 Applications of limited development

Let x_0 be a real number or $-\infty$ or $+\infty$, and f, g are non-zero functions that accepts limited developments in the neighborhood of x_0. We denote by $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ for the coefficients of their regular parts respectively.

3.3.1 Calculation of limits

When calculating the limit $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ and if we obtain one of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$‘, to remove the indeterminacy we develop the functions f and g in the neighborhood of x_0 to the smallest orders m and n, respectively, where $b_n \neq 0$ and $a_n \neq 0$.

Example 3.5

Calculate the limit

$$\lim_{x\to 0}\frac{\ln(1+x)+\frac{1}{2}\sin^2 x-\tan x}{(1-\cos x)\sinh^2 x}.$$

We have

$$\ln(1+x)+\frac{1}{2}\sin^2 x-\tan x=\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4\right)+\frac{1}{2}\left(x-\frac{1}{6}x^3\right)^2-\left(x+\frac{1}{3}x^3\right).$$

$$=-\frac{5}{12}x^4+o(x^4)=-\frac{5}{12}x^4+x^4\varepsilon_1(x) \text{ with } \lim_{x\to 0}\varepsilon_1(x)=0.$$

$$\sinh^2 x\,(1-\cos x)=\left(x+\frac{1}{6}x^3\right)^2\left(1-\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4\right)\right)=\frac{1}{2}x^4+o(x^4)$$

$$=\frac{1}{2}x^4+o(x^4)=\frac{1}{2}x^4+x^4\varepsilon_2(x) \text{ with } \lim_{x\to 0}\varepsilon_2(x)=0.$$

So

$$\lim_{x\to 0}\frac{\ln(1+x)+\frac{1}{2}\sin^2 x-\tan x}{(1-\cos x)\sinh^2 x}=\lim_{x\to 0}\frac{-\frac{5}{12}x^4+x^4\varepsilon_1(x)}{\frac{1}{2}x^4+x^4\varepsilon_2(x)}=\lim_{x\to 0}=\frac{-\frac{5}{12}+\varepsilon_1(x)}{\frac{1}{2}+\varepsilon_2(x)}=-\frac{5}{6}.$$

Example 3.6

Calculate the limit

$$\lim_{x\to 0}\frac{\sin\frac{x^2-x}{x+1}+\ln(1+x)}{\arcsin x-\tan x}$$

We have

$$\frac{x^2-x}{x+1}==-x+2x^2+o(x^2), \text{so } \sin\frac{x^2-x}{x+1}=-x+2x^2+o(x^2)$$

and

$$\sin\frac{x^2-x}{x+1}+\ln(1+x)=-x+2x^2+x-\frac{1}{2}x^2=\frac{3}{2}x^2+x^2\varepsilon_1(x) \text{ with } \lim_{x\to 0}\varepsilon_1(x)=0,$$

$$\arcsin x-\tan x=x+\frac{1}{3}x^3-\left(x+\frac{1}{6}x^3\right)==-\frac{1}{6}x^3+x^3\varepsilon_2(x) \text{ with } \lim_{x\to 0}\varepsilon_2(x)=0.$$

So

$$\lim_{x\to 0}\frac{\sin\frac{x^2-x}{x+1}+\ln(1+x)}{\arcsin x-\tan x}=\lim_{x\to 0}\frac{\frac{1}{2}x^2+x^2\varepsilon_1(x)}{-\frac{1}{6}x^3+x^3\varepsilon_2(x)}=\lim_{x\to 0}\frac{1}{x}\left(\frac{\frac{1}{2}+\varepsilon_1(x)}{-\frac{1}{6}+\varepsilon_2(x)}\right).$$

$$\lim_{\substack{< \\ x\to 0}}\frac{\sin\frac{x^2-x}{x+1}+\ln(1+x)}{\arcsin x-\tan x}=+\infty\,;\ \lim_{\substack{> \\ x\to 0}}\frac{\sin\frac{x^2-x}{x+1}+\ln(1+x)}{\arcsin x-\tan x}=-\infty.$$

3.3.2 Study the relative position of the graph and the tangent line

To determine the relative position of the graph of a function f and its tangent line at the point x_0, we develop the function f in the neighborhood of x_0 to the smallest order n such that $a_n \neq 0$ and $n \geq 2$.

Exercise 3.1

Let the function f be define by $f(x)=\sqrt{x}-\ln\left(\cos\left(\frac{1}{2}x-\frac{1}{2}\right)\right)$, and we denote by $\left(C_f\right)$ the graph representing the function f.

1) Find a limited Development of order 3 in a neighborhood of 1 for the functions f.

2) Deduce the equation of the tangent (T) to the curve $\left(C_f\right)$ at the abscissa point $x=1$.

3) Determine the relative positions of $\left(C_f\right)$ and (T). What do you conclude?.

Solution

1) $F(h)=f(h+x_0)=f(h+1)=\sqrt{1+h}-\ln\left(\cos\frac{h}{2}\right)$.

We have

$$\ln\left(\cos\frac{h}{2}\right)=\ln\left(1+\cos\frac{h}{2}-1\right)=UoV(h) \text{ where } U(h)=\ln(1+h)\text{and } V(h)=\cos\frac{h}{2}-1.$$

Since $\ln(1+x)=h-\frac{1}{2}h^2+\frac{1}{3}h^3+o\left(h^3\right)$ and $\cos\frac{h}{2}-1=-\frac{1}{8}h^2+o\left(h^3\right)$, we get

$$\ln\left(\cos\frac{h}{2}\right)=UoV(h)=\left(-\frac{1}{8}h^2\right)-\frac{1}{2}\left(-\frac{1}{8}h^2\right)^2+\frac{1}{3}\left(-\frac{1}{8}h^2\right)^3=-\frac{1}{8}h^2+o(h^3)$$

And we have $\sqrt{1+h}=1+\frac{1}{2}h-\frac{1}{8}h^2+\frac{1}{16}h^3+o(h^3)$.

So

$$\sqrt{1+h} - ln\left(cos\frac{h}{2}\right) = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3 - \left(-\frac{1}{8}h^2\right) = 1 + \frac{1}{2}h + \frac{1}{16}h^3 + o\left(h^3\right).$$

Substituting $h = x - 1$ we get:

$$f(x) = 1 + \frac{1}{2}(x-1) + \frac{1}{16}(x-1)^3 + o((x-1)^3) = \frac{1}{2}x - \frac{1}{2} + \frac{1}{16}(x-1)^3 + (x-1)^3\,\varepsilon(x).$$

2) We have $f(1) = -\frac{1}{2} = a_0$ so f is differentiable at $x_0 = 1$, and therefore (C_f) accepts tangent line wich we denote by (T). And the equation of (T) is $y = \frac{1}{2}x - \frac{1}{2}$.

3) We have $f(x) - \left(\frac{1}{2}x - \frac{1}{2}\right) = (x-1)^3\left(\frac{1}{16} + \varepsilon(x)\right)$.

If x is sufficiently close to 1, the sign of the difference $f(x) - \left(\frac{1}{2}x - \frac{1}{2}\right)$ is the same sign of $(x-1)^3$, hence the following result:

For $x < 1$, (C_f) is located under the tangent.

For $x > 1$, (C_f) is located above the tangent.

We conclude that (C_f) accepts an inflection point $A_0(1,1)$.

3.3.3 The study of infinite branches of curves

To study the infinite branches and determine the asymptotic lines of the graph (C_f) of function f in the neighborhood of $+\infty$ ($-\infty$, respectively), we develop the function f in the neighborhood of $+\infty$ ($-\infty$, respectively) to the smallest order n, where $a_n \neq 0$ and $n \in \mathbb{N}^*$.

Exercise 3.2

Let the function g be define by $g(x) = xe^{\frac{x-1}{2x^2-3x}}$, and we denote by (C_g) the graph representing the function g.

1) Find a limited Development of order 2 in a neighborhood of ∞ for the functions f.

2) Deduce that the curve (C_g) accepts an asymptote $(\Delta)$and write an equation for it. Study the relative position of (C_g) and (Δ) in a neighborhood of ∞.

Solution

1) We put $G(h) = g\left(\frac{1}{h}\right) = \frac{1}{h}e^{\frac{h-h}{3h-2}}$.

We have

$$\frac{h^2-h}{3h-2}=\frac{1}{2}h+\frac{1}{4}h^2+o(h^2) \text{ and } e^h=1+h+\frac{1}{2}h^2+o(h^2).$$

So

$$G(h)=\frac{1}{h}\left(1+\left(\frac{1}{2}t+\frac{1}{4}t^2\right)+\frac{1}{2}\left(\frac{1}{2}t+\frac{1}{4}t^2\right)^2\right)$$

$$=\frac{1}{h}\left(1+\frac{1}{2}t+\frac{3}{8}t^2+o(h^2)\right).$$

Substituting $h=\frac{1}{x}$ we get:

$$g(x)=x+\frac{1}{2}+\frac{3}{8}\frac{1}{x}+o\left(\frac{1}{x}\right)=x+\frac{1}{2}+\frac{3}{8x}+\frac{1}{x}\varepsilon(x) \text{ with } \lim_{x\to\infty}\varepsilon(x)=0.$$

2) Since $\lim_{x\to\infty}\left(\frac{3}{8x}+\frac{1}{x}\varepsilon(x)\right)=0$, then

a) The line with the equation: $y=x+\frac{1}{2}$ is asymptotic to the curve (C_g) in the neighborhood of ∞.

If x is sufficiently close to 1, the sign of the difference $f(x)-\left(\frac{1}{2}x-\frac{1}{2}\right)$ is the same sign of $(x-1)^3$, hence the following result:

b) We also have: $g(x)-\left(x+\frac{1}{2}\right)=\frac{1}{x}\left(\frac{3}{8}+\varepsilon(x)\right)$, for |x| Big enough, the sign of the difference $g(x)-\left(x+\frac{1}{2}\right)$ is the same sign of $\frac{3}{8}\frac{1}{x}$. Hence the following result:

In a neighborhood of $-\infty$, (C_f) is located under the asymptotic.

In a neighborhood of $+\infty$, (C_f) is located above the asymptotic.

3.4 generalized limited development

Let f be a function defined in the neighborhood of a point 0, - with the possible exception of 0. We suppose that f does not admit a limited development to the neighborhood of 0 but the function $x \to x^\alpha f(x)$ ($\alpha \in \mathbb{R}_+^*$ admits a limited development of order n in the neighborhood of 0. We can then write in the neighborhood of 0 and for $x \neq 0$:

$$x^\alpha f(x)=a_0+a_1x+\ a_2x^2 \dots\dots\dots+a_nx^n+x^n\varepsilon(x) \text{ with } \lim_{x\to 0}\varepsilon(x)=0.$$

From where, the generalized limited development of f in the neighborhood of 0 is:

$$f(x)=\frac{1}{x^\alpha}\left(a_0+a_1x+\ a_2x^2 \dots\dots\dots+a_nx^n+x^n\varepsilon(x)\right) \text{ with } \lim_{x\to 0}\varepsilon(x)=0.$$

3.5 Usual limited developments in 0

Limited Development of usual Functions in 0.

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ \dots\dots\dots + \frac{1}{n!}x^n + o(x^n)$
$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \dots\dots\dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + o(x^{2n+2})$
$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \dots\dots\dots + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n+1})$
$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 \dots\dots\dots + \frac{1}{(2n+1)!}x^{2n+1} + o(x^{2n+2})$
$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots \dots\dots + \frac{1}{(2n)!}x^{2n} + o(x^{2n+1})$
$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ \dots\dots + \frac{\alpha(\alpha-1)\dots\dots(\alpha-n+1)}{n!}x^n + o(x^n) \quad (\alpha \in \mathbb{R}_+^*)$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ \dots\dots\dots + x^n + o(x^n)$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ \dots\dots\dots + (-1)^n x^n + o(x^n)$
$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \ \dots\dots\dots - \frac{1}{n}x^n + o(x^n)$
$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ \dots\dots\dots + \frac{(-1)^{n-1}}{n}x^n + o(x^n)$
$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \dots\dots\dots \frac{(-1)^n}{2n+1}x^{2n+1} + o(x^{2n+2})$
$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\times3}{2\times4}\frac{x^5}{5} + \frac{1\times3\times5}{2\times4\times6}\frac{x^7}{7} \ \dots + \frac{1\times3\times(2n-1)}{2\times4\times(2n)}\frac{x^{2n+1}}{(2n+1)} + o(x^{2n+2})$
$\operatorname{argtanh} x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 \dots\dots\dots \frac{1}{2n+1}x^{2n+1} + o(x^{2n+2})$
$\operatorname{argsinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\times3}{2\times4}\frac{x^5}{5} - \frac{1\times3\times5}{2\times4\times6}\frac{x^7}{7} \ \dots\dots\dots + (-1)^n \frac{1\times3\times(2n-1)}{2\times4\times(2n)}\frac{x^{2n+1}}{(2n+1)} + o(x^{2n+2})$
$\tan x = \ x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + o(x^{12})$
$\tanh x = \ x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \frac{1382}{155925}x^{11} + o(x^{12})$
$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \frac{33}{2048}x^7 - \frac{429}{32768}x^8 + o(x^8)$