Solution of exercise series No 2

Exercise 1

1) Show that the function $y = x + 1 - \frac{1}{3}e^x$, is a solution to the first-order initial value problem $\frac{dy}{dx} = y - x;\ y(0) = \frac{2}{3}$, on $\mathbb{R}$.

we have

$$y = x + 1 - \frac{1}{3}e^x \Longrightarrow \frac{dy}{dx} = 1 - \frac{1}{3}e^x.$$

So

$$\frac{dy}{dx} - y = 1 - \frac{1}{3}e^x - \left(x + 1 - \frac{1}{3}e^x\right) = -x.$$

Olso we have

$$y(0) = 0 + 1 - \frac{1}{3}e^0 = \frac{2}{3}$$

3) give a differential equation whose solutions are of the form:

$$y = \frac{c + x}{x^2 + 1};\ \ c \in \mathbb{R}.$$

we have

$$y = \frac{c + x}{x^2 + 1} \Longrightarrow\ (x^2 + 1)y = c + x$$

$$\Longrightarrow \frac{d}{dx}\left((x^2 + 1)y\right) = \frac{d}{dx}(c + x)$$

$$\Longrightarrow (x^2 + 1)\frac{dy}{dx} + 2xy = 1.$$

Exercise 2

Solve the following separable differential equations:

b) $2x + yy' = 0$.

$$2x + yy' = 0 \Longrightarrow\ \ ydy = -2xdx$$

$$\Longrightarrow\ \int y\ dy = -2\int x\,dx$$

$$\Longrightarrow \frac{1}{2}\, y^2 = -x^2 + c$$

$$\Longrightarrow\; y^2 = -2x^2 + k \;\text{ where }\; k \in \mathbb{R}.$$

We have

$$y(1) = 1 \Longrightarrow\; 1 = -2 + k \;\Longrightarrow\; k = 3.$$

So

$$y^2 = -2x^2 + 3$$

or

$$y = \pm\sqrt{-2x^2 + 3}.$$

c) $xy' + (1 + x)y = 0;\; y(1) = 1.$

$$xy' + (1 + x)y = 0 \Longrightarrow\; \frac{dy}{y} = -\frac{x}{1 + x}dx$$

$$\Longrightarrow\; \int \frac{dy}{y}\, dy = -\int \frac{x}{1 + x}dx == -\int 1 - \frac{1}{1 + x}dx$$

$$\Longrightarrow \ln|y| = -x + \ln|1 + x| + c$$

$$\Longrightarrow y = \pm e^c(1 + x)e^{-x}$$

$$\Longrightarrow\; y = k(1 + x)e^{-x} \;\text{ where }\; k \in \mathbb{R}.$$

We have

$$y(1) = 1 \Longrightarrow\; 1 = k2e^{-1} \;\Longrightarrow\; k = \frac{1}{2}e.$$

So

$$y = \frac{1}{2}(1 + x)e^{1-x}.$$

e) $(4 - x^2)yy' = 2(1 + y^2).$

$$(4 - x^2)yy' = 2(1 + y^2) \Longrightarrow\; \frac{ydy}{1 + y^2} = \frac{2}{4 - x^2}dx$$

$$\Rightarrow \int \frac{ydy}{1+y^2}\ dy = \int \frac{2}{4-x^2}dx$$

$$= \frac{1}{2}\int \frac{1}{2-x} + \frac{1}{2+x}dx$$

$$\Rightarrow \frac{1}{2}\ln(1+y^2) = \frac{1}{2}(-\ln|2-x| + \ln|2+x|) + c$$

$$\Rightarrow \ln(1+y^2) = \ln\left|\frac{2+x}{2-x}\right| + c$$

$$\Rightarrow 1+y^2 = e^c\left|\frac{2+x}{2-x}\right|,$$

$$\Rightarrow 1+y^2 = k\frac{2+x}{2-x} \quad \text{where } \ k \in \mathbb{R}^*.$$

Or

$$\Rightarrow y = \pm\sqrt{k\frac{2+x}{2-x} - 1} \quad \text{where } \ k \in \mathbb{R}^*.$$

Exercise 3

solve the following linear differential equations:

a) $xy' + y = x;\ y(2) = 0.$

We have

$$xy' + y = x \ \Rightarrow \frac{dy}{dy}\ (xy) = x$$

$$\Rightarrow xy = \int x\,dx$$

$$\Rightarrow xy = \frac{1}{2}x^2 + c$$

$$\Rightarrow y = \frac{1}{2}x + \frac{c}{x}.$$

We have

$$y(2) = 0 \ \ \Rightarrow \ \ 0 = \frac{1}{2}2 + \frac{c}{2}$$

$$\Rightarrow \ \ c = -2.$$

So

$$y = \frac{1}{2}x - \frac{2}{x}.$$

c) $y' - 2y = -\frac{2}{1+e^{-2x}}$; $y(0) = 2$.

The integrating factor is

$$v(x) = e^{\int a(x)dx} = e^{\int -2dx} = e^{-2x}.$$

Multiplication of Equation (c) by e^{-2x} gives

$$e^{-2x}y' - 2e^{-2x}y = -\frac{2e^{-2x}}{1+e^{-2x}}$$

or

$$\frac{d}{dx}(e^{-2x}y) = -\frac{2e^{-2x}}{1+e^{-2x}}.$$

Then

$$e^{-2x}y = -\int \frac{2e^{-2x}}{1+e^{-2x}}dx = \ln(1+e^{-2x}) + \mathrm{c}$$

and so

$$y = e^{2x}\ln(1+e^{-2x}) + \mathrm{c}e^{2x}.$$

Since $y(0) = 2$ we have

$$2 = e^0\ln(1+e^0) + \mathrm{c}e^0 \implies c = 2 - \ln 2.$$

Therefore the solution to the initial-value problem is

$$y = e^{2x}\ln(1+e^{-2x}) + (2-\ln 2)e^{2x}.$$

or

$$y = e^{2x}\ln\left(\frac{1}{2} + \frac{1}{2}e^{-2x}\right) + 2e^{2x}.$$

Exercise 4

Solve the following Bernoulli's differential equation:

a) $xy' + 3y = x^2y^2$.

We have

$$xy' + 3y = x^2y^2 \implies y^{-2}\, y' + \frac{3}{x} y^{-1} = x$$

We put $z = y^{1-n} = y^{-1}$ then $z' = -y^{-2}y'$.

Inserting $z = y^{-1}$ and $z' = -y^{-2}y''$ into the differential equation, we get

$$z' - \frac{3}{x} z = -x \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The integrating factor is

$$v(x) = e^{\int a(x)dx} = e^{\int -\frac{3}{x}dx} = \frac{1}{x^3}.$$

Multiplication of Equation (1) by x^3 gives

$$\frac{1}{x^3} z' - \frac{3}{x^4} z = -\frac{1}{x^2}$$

or

$$\frac{d}{dx}\left(\frac{1}{x^3} z\right) = -\frac{1}{x^2}.$$

Then

$$\frac{1}{x^3} z = -\int \frac{1}{x^2} dx = \frac{1}{x} + \mathrm{c}$$

or

$$z = y^{-1} = x^2 + \mathrm{c}x^3.$$

and so

$$y = \frac{1}{x^2 + \mathrm{c}x^3}.$$

c) $y' + 2xy = -xy^4$.

We have

$$y' + 2xy = -xy^4 \implies y^{-4}\, y' + 2xy^{-3} = -x$$

We put $z = y^{1-n} = y^{-3}$ then $z' = -3y^{-4}y'$.

Inserting $z = y^{-3}$ and $z' = -3y^{-4}y''$ into the differential equation, we get

$$-\frac{1}{3}z' + 2xz = -x$$

or

$$z' - 6xz = 3x \dots\dots\dots\dots\dots\dots (2)$$

The integrating factor is

$$v(x) = e^{\int a(x)dx} = e^{\int -6xdx} = e^{-3x^2}.$$

Multiplication of Equation (2) by e^{-3x^2} gives

$$e^{-3x^2}z' - 6xe^{-3x^2}z = 3xe^{-3x^2}$$

or

$$\frac{d}{dx}\left(e^{-3x^2}z\right) = 3xe^{-3x^2}.$$

Then

$$e^{-3x^2}z = -\frac{1}{2}e^{-3x^2} + \mathrm{c}$$

or

$$z = y^{-3} = -\frac{1}{2} + \mathrm{c}e^{3x^2}.$$

or

$$y^3 = \frac{1}{-\frac{1}{2} + \mathrm{c}e^{3x^2}}$$

and so

$$y = \left(\frac{2}{\mathrm{k}e^{3x^2} - 1}\right)^{\frac{1}{3}} \quad \text{where } k \in \mathbb{R} .$$